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Abstract

The Riemann-Lanczos problem for 4-dimensional manifolds was discussed by Bampi and Caviglia. Using exterior differential systems they showed that it was not an involutory differential system until a suitable prolongation was made. Here, we introduce the alternative Janet-Riquier theory and use it to consider the Riemann-Lanczos problem in 2 and 3 dimensions. We find that in 2 dimensions, the Riemann-Lanczos problem is a differential system in involution. It depends on one arbitrary function of 2 independent variables when no differential gauge condition is imposed but on 2 arbitrary functions of one independent variable when the differential gauge condition **is** imposed. For each of the two possible signatures we give the general solution in both instances to show that the occurrence of characteristic coordinates need not affect the result. In 3 dimensions, the Riemann-Lanczos problem is not in involution as a so-called "internal" identity occurs. This does not prevent the existence of singular solutions. A prolongation of this problem, where an integrability condition is added, leads to an involutory prolonged system and thereby generates *non-singular* solutions of the prolonged Riemann-Lanczos problem. We give a singular solution for the unprolonged Riemann-Lanczos problem for the 3-dimensional reduced Gödel spacetime.

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I. Introduction

The problem of generating the spacetime Weyl conformal curvature tensor C_{abcd} from a tensor potential is called *the Weyl-Lanczos problem* and the analogous problem for the Riemann curvature tensor *the Riemann-Lanczos problem*.

The work in this paper is based on the papers [**?**, **?**] and on [**?**] for the Riemann-Lanczos problem as an exterior differential system. But here, for simplicity, we mainly look at the Riemann-Lanczos problem in 2 and 3 dimensions as an exterior differential system and then use Janet-Riquier theory to verify our results. As already explained in many papers such as [**?**, **?**, **?**] we can express the Riemann tensor by means of a third-order tensor potential called the Lanczos tensor with components L_{abc} which are subject to the following symmetries

$$L_{[abc]} = 0 \tag{1}$$

and

$$L_{abc} = L_{[ab]c} \,. \tag{2}$$

If (1) and (2) are imposed, we obtain 2 independent components in 2 dimensions, 8 independent in 3 dimensions and 20 in 4 dimensions. The Riemann-Lanczos equations were first published by Udeschini Brinis [**?**] as

$$R_{abcd} = L_{abc;d} - L_{abd;c} + L_{cda;b} - L_{cdb;a} \,, \tag{3}$$

where ";" denotes covariant differentiation. However, as explained in [**?**, **?**], they did not form a system in involution but had to be prolongated once in order to be in involution. In what follows we include the differential gauge conditions

$$L_{ab}{}^{s}{}_{;s} = 0 \tag{4}$$

but not the trace-free gauge conditions

$$L_{a}{}^{s}{}_{s} = 0 \,. \tag{5}$$

If equations (5) were to be imposed, we would obtain the equation

$$R = 4L^{nk}{}_{k;n} = -4L^{nk}{}_{n;k} = 0$$

for the Ricci scalar R which leads to an inconsistent theory.

II. Janet-Riquier Theory

In the original Janet-Riquier theory of systems of partial differential equations (PDEs), there was an algorithm created to explain how a given system of PDEs could be brought into passive form. Passivity is the *absence* of integrability conditions and orthonomicity is a form of ordering the partial derivatives of a system. The passive orthonomic system of PDEs was the predecessor of what is now called a formally integrable system or involutory system of PDEs.

For a system of partial differential equations it is always an important question whether it possesses a general solution or whether further conditions have to be imposed so that the system is formally integrable or in involution. A theory created by Riquier [?] and developed by Janet [?, ?] and Thomas [?] now called Janet-Riquier theory helps to decide such questions. Applications of the work of Riquier to general relativity have previously been made in papers by Pereira [?, ?, ?][1]. A good account of this theory as in the form as created in [?, ?, ?] is given in Russian by S.P. Finikov [?].

The original approach by Janet, Riquier and Thomas however did not lead to intrinsic results. To achieve this Spencer [?] and then Goldschmidt [?] introduced a new coordinate independent approach based on homological algebra. An account of this theory can be found in [?]. The earlier approach by Thomas [?] was thorough but based on elaborate systems of inequalities for monomials corresponding to a system of PDEs.

There are many algorithms which have been implemented into algebraic computing. Some of them are concerned with the choice of particular rankings such as the REDUCE package DINV by Gerdt [?]. Reid [?] developed a computer package based on MAPLE which brings a system into *solved form* which is a modification of an orthonomic system [?]. Seiler [?, ?] uses the theory of the involutive symbol, which is a modernised version of the original Janet-Riquier theory, to determine whether systems are in involution.

Next, we shall explain a few terms which are important in the original as well as in modernised versions of Janet-Riquier theory. For a system of partial differential equations (=system of PDEs) of order q, which we shall denote by $\mathcal{R}_q$ from now on, we use $x^1, \cdots, x^n$ for the independent variables, where we shall use brackets to indicate powers of any x^i such as in $(x^i)^n$, and u^α for the m dependent variables. Their derivatives on the jet bundle of order q, denoted by $\mathcal{J}^q(\mathbb{R}^n, \mathbb{R}^m)$, are denoted by $u^\alpha_{,J}$, where J is a multi-index. We associate a **monomial** $x^J = (x^1)^{j_1} \cdot ... \cdot (x^n)^{j_n}$ with each partial derivative $u^\alpha_{,J}$, where $J = (j_1, \cdots, j_n)$ is a **multi-index** and $\|J\| = j_1 + \cdots + j_n$ is the order of the derivative. This means that to each set of partial derivatives of each dependent variable u^α corresponds a unique set of monomials.

Then, we **order the partial derivatives** of a system of PDEs in a systematic way. Very often when $x^1, x^2, \cdots, x^n$ are our independent variables, we shall use an inverse lexicographic ordering based on $x^n \succ \cdots \succ x^2 \succ x^1$ on their partial derivatives so that $u_{,n} \succ \cdots \succ u_{,2} \succ u_{,1}$ and so on. Generally, we introduce a **ranking** amongst *all* partial derivatives

Definition 1 *Ranking of Derivatives*
A ranking of derivatives is a total ordering $\mathcal{R}_\succ$ of all the partial derivatives $u^\alpha_{,J}$ (for m, n fixed) satisfying the two conditions (where J and K are multi-indices)
i) If $\|J\| > \|K\|$, then $u^\alpha_{,J} \succ u^\alpha_{,K}$.
ii)If $u^\alpha_{,J^1} \succ u^\alpha_{,J^2}$, then $(u^\alpha_{,J^1})_{,K} \succ (u^\alpha_{,J^2})_{,K}$ for any multi-index K.

A special subclass of these rankings are called **orderly rankings** by which we mean that these are rankings such that i) holds for different indices α referring to different unknowns u^α. A more detailed account on the problem of finding a suitable ranking especially for non-linear systems of equations is given by Rust [?]. Every ranking defined on the partial derivatives induces a ranking amongst the monomials which correspond to the given partial derivatives.

Once a ranking for a system of equations $\mathcal{R}_q$ is determined, the system can be brought into a more organised form called **orthonomic form**. This is achieved by determining the partial derivative highest in the ranking in each equation of $\mathcal{R}_q$ and calling it the **leading** derivative of the equation. Once this equation is solved for its leading derivative which then becomes the only term on its LHS, we call it a **principal** derivative. All other partial derivatives of that order which are not in the set of principal derivatives are called **parametric derivatives**. Based on this we can define an

Definition 2 *Orthonomic System*
A system of partial differential equations $\mathcal{R}_q$ of order q is orthonomic with respect to a given ranking $\mathcal{R}_\succ$ if
i) all the PDEs are solved with respect to their leading derivatives;
ii) no two leading derivatives are the same;
iii) and no parametric derivative in any equation of $\mathcal{R}_q$ can be a principal derivative in another equation of $\mathcal{R}_q$ or even a partial derivative of any order of a principal derivative.

We base our calculations on definition (2) as our own standard. For orthonomic systems the only term on each LHS is the principal derivative so that all derivatives of order q on the RHSs of the equations are parametric derivatives. By differentiating each such equation we can also get the derivatives of the principal derivatives in such a form that we can define **multiplicative variables** for each equation. Those independent variables by means of which we can differentiate the principal derivative of an equation without re-introducing a derivative already produced by means of differentiating another principal derivative are called **multiplicative variables** for the equation. In the same way, once we associate a monomial with each principal derivative, we can define what the multiplicative variables are for that monomial. We say that a variable x^n is multiplicative for the monomials which are of maximal degree in x^n. Further, x^i with $i < n$ is multiplicative for $x^J = (x^1)^{j_1} \cdots (x^n)^{j_n}$ if amongst all monomials of the form $(x^1)^{k_1} \cdots (x^i)^{k_i} (x^{i+1})^{j_{i+1}} \cdots (x^n)^{j_n}$ the monomial $x^J$is **such that** $j_i = \max_k k_i$. This definition given in [?] is based on Janet's original definition [?].

We call a set of monomials *complete* if any multiple can be obtained using multiplicative variables only. We can also define completeness using partial derivatives. Given an orthonomic system, we denote the set of all leading derivatives and their derivatives with respect to multiplicative variables only of the same dependent variable u^α by $\mathcal{L}^\alpha_\succ$. The closure $\bar{\mathcal{L}}^\alpha_\succ$ then includes all derivatives of the derivatives of u^α. We call a system of partial differential equations of order q, $\mathcal{R}_q$, a **complete system** when $\mathcal{L}^\alpha_\succ = \bar{\mathcal{L}}^\alpha_\succ$ for all u^α , $\alpha = 1, \cdots, n$. Once our system is complete, we can also ask whether it is *passive*. A system of PDEs having a complete set of monomials is called **passive** if any computation of a principal derivative is equivalent to any computation obtained using multiplicative variables only. Otherwise, the additional equations have to be added as integrability conditions. If a complete system of PDEs is given in orthonomic form, we speak of a

Definition 3 *Passive Orthonomic System*
An orthonomic system of equations $\mathcal{R}_q$ is passive with respect to a ranking $\mathcal{R}_\succ$, if the sets of all leading derivatives $\mathcal{L}^\alpha_\succ$ with respect to the chosen ranking $\mathcal{R}_\succ$ are complete and **no integrability conditions** *occur.*

Riquier [?] formulated an important theorem for the local existence of analytic solutions based on this, namely,

Theorem 1 *Riquier*
Given a passive system with respect to an orderly ranking $\mathcal{R}_{\succ}$, then its formal power series solution converges.

The main difficulty which arises in Janet-Riquier theory is that so many results are not intrinsic and depend on the choice of a ranking. In a more recent version this problem is re-examined by introducing special coordinates called **δ-regular coordinates** and by using the theory of the **involutive symbol** of a system $\mathcal{R}_q$ which we shall introduce next. But before that, we shall discuss a Janet example in detail in order to illustrate the theory exhibited so far.

A classical example of Janet [**?**, **?**, **?**] consists of two second-order equations with 3 independent variables x^1, x^2, x^3 and one dependent variable u so that $n = 3$, $m = 1$ and $q = 2$. The two equations are

$$\begin{aligned} u_{,33} - x^2 u_{,11} &= 0\,, \\ u_{,22} &= 0\,. \end{aligned} \tag{6}$$

If we impose the coordinate ranking $x^3 \succ x^2 \succ x^1$, we obtain the ranking $u_{,33} \succ u_{,23} \succ u_{,22} \succ u_{,13} \succ u_{,12} \succ u_{,11}$ amongst the second-order partial derivatives. The leading derivatives then are $u_{,33}$ in the first equation and $u_{,22}$ in the second equation so that the system in orthonomic form is given by

$$\begin{array}{ll} u_{,33} = x^2 u_{,11} & x^1\ x^2\ x^3 \\ u_{,22} = 0 & x^1 x^2 \bullet\ . \end{array} \tag{7}$$

The principal derivatives are $u_{,33}$ and $u_{,22}$ which have the corresponding set of monomials $\{(x^3)^2, (x^2)^2\}$ and the parametric derivatives are $u_{,11}, u_{,12}, u_{,13}$, $u_{,23}$. In the first equation all variables x^1, x^2, x^3 are multiplicative variables but only x^1 and x^2 in the second equation whereas the variable x^3 is not a multiplicative variable in the second equation (which is indicated by a dot). This is because, by means of differentiating the first equation twice with respect to x^2, $u_{,2233}$ and higher derivatives can already be created[1]. For the Janet example (6), its completion is given in [**?**],[**?**]. When choosing the above ranking based on $x^3 \succ x^2 \succ x^1$, we obtain for $\mathcal{L}_{\succ}$ and for $\bar{\mathcal{L}}_{\succ}$

$$\mathcal{L}_{\succ} = \{u_{,33}; u_{,22}; u_{,133}; u_{,233}; u_{,333}; u_{,122}; u_{,222}\}$$

$$\bar{\mathcal{L}}_{\succ} = \{u_{,33}; u_{,22}; u_{,133}; u_{,233}; u_{,333}; u_{,122}; u_{,222}; u_{,223}\}\,.$$

Therefore, $u_{,223}$ cannot be produced using multiplicative variables only - a fact reflected by the impossibility of producing the monomial $(x^2)^2 \cdot x^3$ using the set $\{(x^3)^2, (x^2)^2\}$. We have to add $u_{,223}$ so that the new completed system is

$$\begin{array}{lll} u_{,223} = & 0 & x^1\ x^2\ \bullet \\ u_{,33} - x^2 u_{,11} = & 0 & x^1\ x^2\ x^3 \\ u_{,22} = & 0 & x^1\ x^2\ \bullet\ . \end{array}$$

But this system is not passive because, using multiplicative variables only, we can form $u_{,2233} - x^2 u_{,1122} - 2u_{,112} = 0$ which leads to the integrability condition $u_{,112} = 0$. We refer the reader to [**?**] for a detailed discussion of the completion process and only state the final prolonged complete system as

$$\begin{array}{lll}
u_{,11113} = & 0 & x^1 \bullet \bullet \\
u_{1123} = & 0 & x^1 \bullet \bullet \\
u_{,1111} = & 0 & x^1 \bullet \bullet \\
u_{,223} = & 0 & x^1\, x^2 \bullet \\
u_{,112} = & 0 & x^1 \bullet \bullet \\
u_{,33} - x^2 u_{,11} = 0 & & x^1\, x^2\, x^3 \\
u_{,22} = & 0 & x^1\, x^2 \bullet ,
\end{array}$$

where *prolongation* of a system of equations $\mathcal{R}_q$ simply means adding all derivatives of order $q+1$ of the equations in $\mathcal{R}_q$ leading to the prolonged system of equations $\mathcal{R}_{q+1}$, a process which can be carried out several times if necessary.

A. Involutive Symbol and Formal Integrability

In modern versions following Spencer [**?**] one wishes to replace the coordinate-dependent theory of complete passive orthonomic systems of equations $\mathcal{R}_q$ by means of quantities which will no longer depend on a choice of coordinates. A theoretical approach based on map diagrams from homological algebra [**?**, **?**] also explained in [**?**] does fulfil these requirements.

For practical purposes though we shall rely on the theory of the **involutive symbol** and on **δ-regular coordinates**. The **symbol** of a system of PDEs involves only the highest-order partial derivatives of each equation in $\mathcal{R}_q$ which simplifies calculations especially of large systems significantly. We denote the quantities corresponding to each partial derivative $u^\alpha_{,J}$ by V^α_J and define the symbol $\mathcal{M}_q$ as

Definition 4 *Symbol of $\mathcal{R}_q$*
A system of partial differential equations $\mathcal{R}_q$ of order q locally described by p equations in solved form as $\Phi^\tau(x^i, u^\alpha, u^\alpha_{,J}) = 0$ for $\tau = 1, ..., p$, has a solution space $\mathcal{M}_q$ for the unknowns V^α_J with $\alpha = 1, ..., m$, $\|J\| = q$:

$$\mathcal{M}_q : \sum_{\alpha, \|J\|=q} \left(\frac{\partial \Phi^\tau}{\partial u^\alpha_{,J}} \right) V^\alpha_J = 0 \,, \tag{8}$$

where we formally differentiate with respect to the $u^\alpha_{,J}$. $\mathcal{M}_q$ is called the **symbol of** *$\mathcal{R}_q$.*

For simplicity the matrix rather than the map is usually regarded as the symbol of $\mathcal{R}_q$. We associate with each symbol equation its multiplicative variables which are the same as the multiplicative variables its corresponding equation in $\mathcal{R}_q$ adopts. Once each symbol equation has its multiplicative variables allocated, we determine the **class of an equation** in $\mathcal{M}_q$ by counting the number of multiplicative variables it adopts - a number denoted by k such that $0 \leq k \leq n$. We carry this out with all equations occurring in $\mathcal{M}_q$, counting how many equations of each class there are. So we define

$$\beta_q^{(k)} := \text{number of equations of class } k \text{ in } \mathcal{M}_q \ (\text{or} \mathcal{R}_q) \,.$$

The definition of the **Cartan characters** [2] $\alpha_q^{(k)}$is based on the $\beta_q^{(k)}$ and given by

$$\alpha_q^{(k)} := m \cdot \binom{n+q-k-1}{q-1} - \beta_q^{(k)} \,, \tag{9}$$

where $m \cdot \sum_{k=1}^{n} \binom{n+q-k-1}{q-1}$ is the total number of partial derivatives of order q that a system $\mathcal{R}_q$ will have. The Cartan character $\alpha_q^{(k)}$ represents the number of remaining independent partial derivatives of order q in the subsystem of class k after the removal of the number of the principal derivatives of class k in that subsystem. In δ-regular coordinates $\alpha_q^{(k)}$ equals the number of parametric derivatives of class k and order q.

Because the notion "class of an equation" is **ranking-dependent**, we must ensure that we obtain intrinsic results. This can be achieved when we use a ranking for which we obtain the maximal possible value for $\beta_q^{(n)}$ and then for $\beta_q^{(n-1)} + \beta_q^{(n)}$, and so on until we obtain the maximal possible value for $\sum_{k=1}^{n} \beta_q^{(k)}$. This is equivalent to obtaining the minimal possible value for $\alpha_q^{(n)}$, then for $\alpha_q^{(n-1)} + \alpha_q^{(n)}$ and so on until the minimal possible value for $\sum_{k=1}^{n} \alpha_q^{(k)}$ is achieved. In an arbitrarily given coordinate system, a prolongation to higher order is sometimes necessary to obtain intrinsic results. This can be avoided by performing a coordinate transformation, where a linear coordinate transformation is sufficient [**?**], and by checking again for minimal and maximal values respectively of the $\alpha_q^{(k)}$ and the $\beta_q^{(k)}$ in the above sense. Once this is fulfilled, we are using a system of **δ-regular coordinates**.

It is important to determine whether a given system of equations $\mathcal{R}_q$ possesses identities or not. When no identities are present, the symbol is said to be involutive which can be equivalently expressed as [**?**]

Theorem 2 *Involutive Symbol*
In a δ-regular coordinate system the following conditions are equivalent:
i) The symbol $\mathcal{M}_q$ is involutive ;
ii) dim $(\mathcal{M}_{q+1}) = \sum_{k=1}^{n} k \cdot \alpha_q^{(k)}$;
iii) for the rank r of $\mathcal{M}_{q+1}$ it is $r(\mathcal{M}_{q+1}) = \sum_{k=1}^{n} k \cdot \beta_q^{(k)}$;
iv) prolongation with respect to **non-multiplicative variables** *does not lead to any new equations.*

But a system of equations $\mathcal{R}_q$ which has an involutive symbol $\mathcal{M}_q$ can still admit integrability conditions. They can be revealed by means of projecting our prolonged system $\mathcal{R}_{q+1}$, which is obtained by differentiating $\mathcal{R}_q$ with respect to all its n independent variables, back onto $\mathcal{R}_q$. In general, we shall denote first projections onto lower-order systems by $\mathcal{R}_{q+r}^{(1)} = \pi_{q+r}^{(q+r+1)}(\mathcal{R}_{q+r+1})$. If $\mathcal{R}_q^{(1)}$ is not identical to $\mathcal{R}_q$, then integrability conditions occur, but if they are identical we can characterise a formally integrable system $\mathcal{R}_q$ as [**?**, **?**]

Definition 5 *Formal Integrability*
A system of partial differential equations $\mathcal{R}_q$ is formally integrable means that $\mathcal{R}_{q+r}^{(1)} = \mathcal{R}_{q+r}$ for all $r \geq 0$.

A special situation occurs when $\mathcal{M}_q$ is involutive *and* $\mathcal{R}_q$ is formally integrable [**?**]. Then, we obtain ideal systems of equations which are called

Definition 6 *Involutive Systems of Equations*
A system of equations $\mathcal{R}_q$ is involutive if and only if its symbol $\mathcal{M}_q$ is involutive and $\mathcal{R}_q$ is formally integrable.

Note that if we are dealing with a vector bundle, it is always valid $\dim(\mathcal{R}_q^{(1)}) = \dim(\mathcal{R}_{q+1}) - \dim(\mathcal{M}_{q+1})$. This means that when we know the dimension of the prolonged system of equations $\mathcal{R}_{q+1}$ and the dimension of its prolonged symbol $\mathcal{M}_{q+1}$, we automatically know the dimension of the system of equations projected back onto order q called $\mathcal{R}_q^{(1)}$. This can be a great help in deciding whether a system has integrability conditions.

The subsequent theorems, which can be found in [**?**, **?**], are very useful for practical applications as only a limited number of steps need to be carried out to check for involutivity. First we look at

Theorem 3 *Let $\mathcal{M}_q$ be an involutive symbol of a system of equations $\mathcal{R}_q$, then*
i) the symbol $\mathcal{M}_{q+1}$ is also involutive, and
ii) $(\mathcal{R}_q^{(1)})_{+1} = \mathcal{R}_{q+1}^{(1)}$.

Here i) states that, for any involutive symbol $\mathcal{M}_q$, a prolongation is trivial and the prolonged symbol is again involutive and (ii) says that in this case projections and prolongations commute. A proposition following from this is

Theorem 4 *Proposition*
Given a system of differential equations $\mathcal{R}_q$, where the number of equations equals the number of dependent variables m, then, $\mathcal{R}_q$ has either identities or integrability conditions which implies that $\beta_q^{(n-1)} > 0$.

Based on the previous definitions, we would have to perform infinitely many prolongations in order to decide whether a given system $\mathcal{R}_q$ is involutive or not. Fortunately this is not necessary and we can use the important

Theorem 5 *Criterion for Involution*
$\mathcal{R}_q$ is a system in involution means that its symbol $\mathcal{M}_q$ is involutive <u>*and*</u> *$\mathcal{R}_q^{(1)} = \mathcal{R}_q$.*

We shall now present the modern version of Janet-Riquier theory as given above using the Janet example (6). Its symbol $\mathcal{M}_2$ is given by

$$\begin{array}{lll|l} V_{33} = x^2 V_{11} & & x^1\ x^2\ x^3 & 3 \\ V_{22} = 0 & & x^1\ x^2\ \bullet & 2, \end{array}$$

where the integers on the RHS of each equation indicate the class of this equation. This leads to $\beta_2^{(1)} = 0, \beta_2^{(2)} = \beta_2^{(3)} = 1$ and so $\sum_{k=1}^{3} k \cdot \beta_2^{(k)} = 5$ which gives us the **total number of multiplicative variables** for $\mathcal{M}_2$ to be 5. In order for $\mathcal{M}_2$ to be involutive, we have to verify that $r(\mathcal{M}_3)$ is equal to 5. Firstly, we determine the new prolonged system of equations $\mathcal{R}_3$ which consists of the 8 equations

$$\begin{array}{ll} \boxed{1} & 0 = u_{,33} - x^2 u_{,11} \\ \boxed{2} & 0 = u_{,22} \\ \boxed{3} & 0 = u_{,133} - x^2 u_{,111} \\ \boxed{4} & 0 = u_{,233} - x^2 u_{,112} - u_{,11} \\ \boxed{5} & 0 = u_{,333} - x^2 u_{,113} \\ \boxed{6} & 0 = u_{,122} \\ \boxed{7} & 0 = u_{,222} \\ \boxed{8} & 0 = u_{,223}\ . \end{array}$$

Next, we determine $\dim(\mathcal{R}_3)$, where $\dim(\mathcal{R}_q)$ is defined as the total number of dependent variables and all their partial derivatives up to order q **minus** the number of independent equations which constitute $\mathcal{R}_q$. In this way, we can easily see that $\dim(\mathcal{R}_3) = 20-8 = 12$, where 8 is the total number of independent equations in $\mathcal{R}_3$. The number 20 stands for $20 = 1+3+6+10$ for one unknown u, 3 first-order partial derivatives $u_{,1}, u_{,2}, u_{,3}$, 6 second-order partial derivatives $u_{,11}, u_{,12}, u_{,13}, u_{,22}, u_{,23}, u_{,33}$ and 10 third-order partial derivatives $u_{,111}, u_{,112}, u_{,113}, u_{,122}, u_{,123}, u_{,133}, u_{222}, u_{223}, u_{,233}, u_{,333}$. Next, we determine $\mathcal{M}_3$ which is given in orthonomic form as

$$\begin{array}{lll|l} V_{333} = x^2 V_{113} & & x^1\; x^2\; x^3 & 3 \\ V_{233} = x^2 V_{112} & & x^1\; x^2\; \bullet & 2 \\ V_{223} = 0 & & x^1\; x^2\; \bullet & 2 \\ V_{222} = 0 & & x^1\; x^2\; \bullet & 2 \\ V_{133} = x^2 V_{111} & & x^1\; \bullet\; \bullet & 1 \\ V_{122} = 0 & & x^1\; \bullet\; \bullet & 1, \end{array}$$

where we can see that there are 2 equations of class 1, 3 equations of class 2 and 1 equation of class 3 resulting in $\beta_3^{(1)} = 2, \beta_3^{(2)} = 3, \beta_3^{(3)} = 1$. This leads to $\sum_{k=1}^{3} k\cdot\beta_3^{(k)} = 1\cdot 2+2\cdot 3+3\cdot 1 = 11$ being the total number of multiplicative variables which $\mathcal{M}_3$ adopts. We also find that $r(\mathcal{M}_3) = 6$ which is not equal the total number of multiplicative variables $\mathcal{M}_2$ adopts, which is 5, and therefore $\mathcal{M}_3$ is not involutive. We need to prolongate $\mathcal{R}_3$ further to $\mathcal{R}_4$ and we refer the reader to [**?**] for further details subsequent calculations. We only state here that during this process the two integrability conditions

$$\begin{aligned} u_{,112} &= 0 \\ u_{,1111} &= 0 \end{aligned} \tag{10}$$

occur. Finally, we obtain the result that the prolonged fifth-order system, which is augmented by (10), $\mathcal{R}_5^{(2)}$, forms a system in involution and the Janet algorithm terminates.

III. The Riemann-Lanczos Problem in 2 Dimensions

We shall apply the theory of EDS and Janet-Riquier theory in order to examine the Riemann-Lanczos problems in 2 and in 3 dimensions. For a review of the theory of EDS see [**?**, **?**, **?**, **?**, **?**, **?**].

We show that the Riemann-Lanczos problem is in involution with respect to its two independent variables for both possible choices of signature and show that the occurrence of characteristic coordinates need not affect the result. We shall look at the problem first as an exterior differential system and then as a system of partial differential equations (systems of PDEs) applying Janet-Riquier theory.

A. The Riemann-Lanczos Problem in 2 Dimensions as an EDS

Firstly, we consider the Riemann-Lanczos problem in a 2-dimensional spacetime. We know that in 2 spacetime dimensions, there is only one independent component of the Riemann tensor and therefore only one independent

Riemann-Lanczos equation. We now choose $(x^1, x^2, L_{121}, L_{122}, P_{1211}, P_{1212}, P_{1221}, P_{1222})$ as local coordinates on the jet bundle $\mathcal{J}^1(\mathbb{R}^2, \mathbb{R}^2)$ so that it is 8-dimensional. We leave the metric tensor components completely arbitrary so that the following results will hold for any 2-dimensional spacetime. The only independent Riemann-Lanczos equation is given in solved form as

$$\begin{aligned} f^{(R)}_{1212} &= R_{1212} - 2P_{1212} + 2P_{1221} + 2L_{121}(\Gamma^1_{12} - \Gamma^2_{22}) - 2L_{122}(\Gamma^1_{11} + \Gamma^2_{12}) \\ &= 0 \, . \end{aligned} \tag{11}$$

We define α_{1212e} in the 2-dimensional case to be

$$\alpha_{1212e} = R_{1212,e} - 2L_{121}(\Gamma^1_{12,e} - \Gamma^2_{22,e}) - 2L_{122}(\Gamma^1_{11,e} + \Gamma^2_{12,e}) \tag{12}$$

where $e = 1, 2$. The Pfaffian system derived from (11) is locally given by

$$\begin{aligned} \theta^1 &= \alpha_{12121} dx^1 + \alpha_{12122} dx^2 - 2dP_{1212} + 2dP_{1221} + 2(\Gamma^1_{12} - \Gamma^2_{22}) dL_{121} \\ &\quad -2(\Gamma^1_{11} + \Gamma^2_{12}) dL_{122} \, , \\ \theta^2 &= dL_{121} - P_{1211} dx^1 - P_{1212} dx^2 \, , \\ \theta^3 &= dL_{122} - P_{1221} dx^1 - P_{1222} dx^2 \, , \\ d\theta^2 &= dx^1 \wedge \ P_{1211} + dx^2 \wedge P_{1212} \, , \\ d\theta^3 &= dx^1 \wedge \ P_{1211} + dx^2 \wedge P_{1212} \, , \end{aligned} \tag{13}$$

where θ^1 is the exterior derivative of $f^{(R)}_{abcd}$ and θ^2, θ^3 are the two contact conditions and $d\theta^2, d\theta^3$ their exterior derivatives. The zeroth character is $s_0 = 3$, since we only have to count the number of 1-forms in (13). Omitting all the terms involving dx^1, dx^2 in the 1-forms in 13, the same number is obtained so that $s'_0 = 3$ leading to our first result $s_0 = s'_0 = 3$. In order to obtain s_1, we must form the first polar element $H((E^1)_x)$ of a 1-dimensional integral element $(E^1)_x$ formed by the Vessiot vector field U

$$U = U^e \frac{\partial}{\partial x^e} + P_{121e} U^e \frac{\partial}{\partial L_{121}} + P_{122e} U^e \frac{\partial}{\partial L_{122}} + U_{121e} \frac{\partial}{\partial P_{121e}} + U_{122e} \frac{\partial}{\partial P_{122e}} \, , \tag{14}$$

where

$$U_{1212} = U_{1221} + \delta_1 U^1 + \delta_2 U^2 \tag{15}$$

because

$$\begin{aligned} \delta_1 &= \frac{1}{2} \alpha_{12121} + (\Gamma^1_{12} - \Gamma^2_{22}) P_{1211} - (\Gamma^1_{11} + \Gamma^2_{12}) P_{1221} \, , \\ \delta_2 &= \frac{1}{2} \alpha_{12122} + (\Gamma^1_{12} - \Gamma^2_{22}) P_{1212} - (\Gamma^1_{11} + \Gamma^2_{12}) P_{1222} \, . \end{aligned} \tag{16}$$

This means that for the Vessiot vector field U the components $U^1, U^2, U_{1211}, U_{1221}, U_{1222}$ can be chosen arbitrarily and so we can form $H((E^1)_x)$. For the coefficient matrix of the second polar system $H((E^2)_x)$, we obtain

equation	dx^1	dx^2	dL_{121}	dL_{122}	dP_{1211}	dP_{1212}	dP_{1221}	dP_{1222}
$df^{(R)}_{1212}$	α_{12121}	α_{12122}	$2(\Gamma^1_{12} - \Gamma^2_{22})$	$-2(\Gamma^1_{11} + \Gamma^2_{12})$	0	-2	2	0
θ^1	$-P_{1211}$	$-P_{1212}$	1	0	0	0	0	0
θ^2	$-P_{1221}$	$-P_{1222}$	0	1	0	0	0	0
$2(U\rfloor d\theta^1)$	$-U_{1211}$	$-U_{1212}$	0	0	U^1	U^2	0	0
$2(U\rfloor d\theta^2)$	$-U_{1221}$	$-U_{1222}$	0	0	0	0	U^1	U^2
$2(V\rfloor d\theta^1)$	$-V_{1211}$	$-V_{1212}$	0	0	V^1	V^2	0	0
$2(V\rfloor d\theta^2)$	$-V_{1221}$	$-V_{1222}$	0	0	0	0	V^1	V^2

Here, U and V are the first and second Vessiot vector fields to be chosen. From the first 5 rows and their reduced counterparts of the above Matrix we determine s_1 and s'_1. We can easily see that the rank of this 5x8-Matrix is 5 and the rank for the reduced matrix, where we just omit the first 2 columns for dx^1 and dx^2, is 5 as well. Therefore, the Cartan character s_1, s'_1 are $s_1 = s'_1 = 5 - 3 = 2$. Apart from the same conditions that already hold for U, the second Vessiot vector field V must also satisfy the additional conditions

$$V^e U_{abce} - U^e V_{abce} = 0\,. \tag{17}$$

For V_{1212}, the same conditions apply as for U_{1212} so that

$$V_{1212} = V_{1221} + \delta_1 V^1 + \delta_2 V^2 \tag{18}$$

with δ_1, δ_2 as defined above. The only free components left for V are V^1, V^2, V_{1221}. In order to obtain the rank of the second polar element, we need to consider all rows of the above matrix. First, we look at the reduced 6x7-matrix therefore omitting the first two columns under dx^1 and dx^2. Let us denote the i^{th} row by (i), $i = 1, \cdots, 7$ and check whether the 7 rows are linearly dependent by looking for possible linear relations among them. We find

$$(1) = A_1 \cdot (2) + A_2 \cdot (3) + B_1 \cdot (4) + B_2 \cdot (5) + C_1 \cdot (6) + C_2 \cdot (7)\,, \tag{19}$$

where

$$\begin{array}{ll} A_1 = 2(\Gamma^1_{12} - \Gamma^2_{22})\,, & A_2 = -2(\Gamma^1_{11} - \Gamma^2_{12})\,, \\ B_1 = 2\frac{V^1}{\delta}\,, & B_2 = 2\frac{V^2}{\delta}\,, \\ C_1 = -2\frac{U^1}{\delta}\,, & C_2 = -2\frac{U^2}{\delta}\,, \end{array}$$

and $\delta = U^1V^2 - U^2V^1$. This leads to the result $s'_2{=}6 - 3 - 2 = 1$. Now, we must check whether or not the same linear combination occurs amongst the 7 rows for the full polar system:

$$(1) = A_1 \cdot (2) + A_2 \cdot (3) + B_1 \cdot (4) + B_2 \cdot (5) + C_1 \cdot (6) + C_2 \cdot (7)\,, \tag{20}$$

which must also hold for the **first two columns**. This leads to the two conditions

$$\begin{aligned} \alpha_{12121} = {} & -A_1 \cdot P_{1211} - A_2 \cdot P_{1221} - B_1 \cdot U_{1211} - B_2 \cdot U_{1221} - C_1 \cdot V_{1211} \\ & -C_2 \cdot V_{1221}\,, \\ \alpha_{12122} = {} & -A_1 \cdot P_{1212} - A_2 \cdot P_{1222} - B_1 \cdot U_{1212} - B_2 \cdot U_{1222} - C_1 \cdot V_{1212} \\ & -C_2 \cdot V_{1222}\,, \end{aligned} \tag{21}$$

where we have to insert the above known expressions for $U_{1212}, V_{1212}, V_{1211}$ and V_{1222} and it is not permitted to restrict any of the U^e, V^e. By inserting all the above determined values for the A_i, B_i, C_i, we see that in the linear relations (21) the free components $U_{1211}, U_{1221}, U_{1222}, V_{1221}$ cancel out and both relations in (21) hold identically without imposing any further restrictions. Therefore, the same linear combination (20) holds for the full system. The second character and its reduced counterpart therefore coincide so that $s_2 = s'_2 = 1$ and the involutive genus g is $g = \sum_{i=1}^2 s_i = 2$.

We can also compute the characters using the tableau and by determining the torsion we then find out whether the system possesses integrability conditions. Given a Pfaffian system $\mathcal{P}$ consisting of s 1-forms θ^α with independence condition $\Omega = \omega^1 \wedge \cdots \wedge \omega^3$, we denote by π^λ all the extra forms such that $(\theta^\alpha, \omega^i, \pi^\lambda)$, $1 \le \alpha \le s$, form a coframe on our formally N-dimensional manifold $\mathcal{M}$. We can then write

$$d\theta^\alpha = A^\alpha_{\lambda i} \wedge \omega^i + \frac{1}{2} B^\alpha_{ij} \omega^i \wedge \omega^j + \frac{1}{2} C^\alpha_{\lambda\kappa} \pi^\lambda \wedge \pi^\kappa \,(\mathrm{mod}\, \mathcal{I}(\mathcal{P}))\,. \tag{22}$$

In equations (22), the $A^\alpha_{\lambda i}$ form the **tableau matrix** and the B^α_{ij} are called the **torsion terms**. Note that if the coefficients $C^\alpha_{\lambda\kappa} = 0$, then the system is said to be **quasi-linear**. In order to form a complete coframe for our example (13), we see that we have to add three 1-forms π^Λ, where Λ now is a *collective index* with $\Lambda \in \{1211, 1221, 1222\}$, to the five 1-forms $\theta^1, \theta^2, \theta^3, \omega^1, \omega^2$ and so we choose

$$\pi_{1211} := dP_{1211}, \;\; \pi_{1221} := dP_{1221}, \;\; \pi_{1222} := dP_{1222}\,.$$

The dP_{1212} can be expressed through θ^1 as

$$\begin{aligned} dP_{1212} = \pi_{1221} + (\frac{1}{2}\alpha_{12121} + (\Gamma^1_{12} - G^2_{22})(P_{1211} - (\Gamma^1_{11} + \Gamma^2_{12})P_{1221})\omega^1 + (\frac{1}{2} \\ \alpha_{12122} + (\Gamma^1_{12} - G^2_{22})(P_{1212} - (\Gamma^1_{11} + \Gamma^2_{12})P_{1222})\omega^2 \,(\mathrm{mod}\,\{\theta^\alpha\}) \end{aligned} \tag{23}$$

Having accomplished this, we can then write the exterior derivatives of the contact conditions as

$$\begin{aligned} d\theta^1 &\equiv 0 \\ d\theta^2 &\equiv -\pi_{1211} \wedge \omega^1 - \pi_{1221} \wedge \omega^2 + (\frac{1}{2}\alpha_{12121} + (\Gamma^1_{12} - \Gamma^2_{22})P_{1211} \\ &\quad -(\Gamma^1_{11} + \Gamma^2_{12})P_{1221})\omega^1 \wedge \omega^2 \\ d\theta^3 &\equiv -\pi_{1221} \wedge \omega^1 - \pi_{1222} \wedge \omega^2\,. \end{aligned} \tag{24}$$

Therefore, we obtain for the tableau matrices

$$A^\alpha_{\Lambda 1} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \; A^\alpha_{\Lambda 2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,,$$

where $\alpha = 1, 2, 3$ and Λ is one of the collective indices $\Lambda \in \{1211, 1221, 1222\}$. This leads to $s'_1 = 2$, $s'_2 = 1$, and, the only non-vanishing torsion term is given by

$$B^2_{12} = -[\frac{1}{2}\alpha_{12121} - (\Gamma^1_{12} - \Gamma^2_{22})P_{1211} + (\Gamma^1_{11} + \Gamma^2_{12})P_{1221}]\,. \tag{25}$$

If we wish to absorb the torsion coefficients, we must find a transformation Φ with $\pi^\lambda \;\rightarrow\; \pi^\lambda + p^\lambda_i \omega^i$ and quantities p^λ_i such that

$$0 = \tilde{B}^\alpha_{ij} = B^\alpha_{ij} + A^\alpha_{\lambda j} p^\lambda_i - A^\alpha_{\lambda i} p^\lambda_j\,.$$

In our case this leads to the system

$$\begin{aligned} 0 &= A^1_{\Lambda 2} p^\Lambda_1 - A^1_{\Lambda 1} p^\Lambda_2 \\ 0 &= B^2_{12} + A^2_{\Lambda 2} p^\Lambda_1 - A^2_{\Lambda 1} p^\Lambda_2 \\ 0 &= A^3_{\Lambda 2} p^\Lambda_1 - A^3_{\Lambda 1} p^\Lambda_2\,. \end{aligned} \tag{26}$$

One solution *Ansatz* to fulfil (26) is to choose $p^2_1 := B^2_{12}$ while all other p^Λ_i vanish: thus the torsion is absorbed and the system is therefore in involution [**?**, **?**].

Adding the exterior derivative of the differential gauge condition $L_{12}{}^{s}{}_{;s} = 0$, which is locally given by

$$dL_{12}{}^{s}{}_{;s} = g^{11}dP_{1211} + g^{12}(dP_{1221} + dP_{1212}) + g^{22}dP_{1222} - \delta_3 dL_{121} - \delta_4 dL_{122} \\ +\delta_5 dx^1 + \delta_6 dx^2 \,, \tag{27}$$

where $\delta_3, \cdots, \delta_6$ are defined in Appendix A, does not change the results qualitatively. We can write down the polar systems again and from this obtain the polar matrices of which the matrix of $H(E^2)_x$ is given in Appendix A.

Obviously $s_0 = s_0' = 4$ and we find that no linear combinations are possible amongst the first 6 rows or their reduced counterparts so that $s_1 = s_1' = 2$. But for s_2' there exist multipliers A_i, B_j such that

$$(7) = A_1 \cdot (1) + A_2 \cdot (2) + A_3 \cdot (3) + A_4 \cdot (4) + A_5 \cdot (5) + A_6 \cdot (6)\,, \\ (8) = B_1 \cdot (1) + B_2 \cdot (2) + B_3 \cdot (3) + B_4 \cdot (4) + B_5 \cdot (5) + B_6 \cdot (6) \tag{28}$$

now with multipliers $A_1, \cdots, A_6$ and $B_1, \cdots, B_6$ as given in Appendix A. In order for these linear combinations to hold for the full rows (7) and (8) as well, the following 4 equations have to hold:

$$V_{1211} = -A_1\alpha_{12121} - A_2\delta_5 + A_3P_{1211} + A_4P_{1221} + A_5U_{1211} + A_6 \\ U_{1221}, \tag{29}$$

$$V_{1221} = -B_1\alpha_{12121} - B_2\delta_5 + B_3P_{1211} + B_4P_{1221} + B_5U_{1211} + B_6 \\ U_{1221}, \tag{30}$$

$$V_{1212} = -A_1\alpha_{12122} - A_2\delta_6 + A_3P_{1212} + A_4P_{1222} + A_5U_{1212} + A_6 \\ U_{1222}, \tag{31}$$

$$V_{1222} = -B_1\alpha_{12122} - B_2\delta_6 + B_3P_{1212} + B_4P_{1222} + B_5U_{1212} + B_6 \\ U_{1222}, \tag{32}$$

where U is the first Vessiot vector field and V the second one. After a lengthy calculation of which details are given in [?] we find that with (18) the above expressions (29) to (32) hold. Therefore, the full rows (7) and (8) are again linearly dependent which leads to $s_2 = s_2' = 0$ and $g = 2$. Note that all the above results can be verified using a REDUCE code given in [?].

B. The 2-dimensional Riemann-Lanczos Problem as a System of PDEs

The Riemann-Lanczos problem in 2 dimensions is an example of a system of partial differential equations for which applying Janet-Riquier theory is much more economical. We shall also see in Section C below that it is of great importance to compute the characters using δ-regular coordinates. There, if characteristic coordinates are used, they will lead to non-intrinsic results so that a coordinate transformation will then have to be carried out. We have a system of equations $\mathcal{R}_1$ with $n = m = 2, q = 1$ and the symbol $\mathcal{M}_1$ consists only of a single equation

$$-2V_{1212} + 2V_{1221} = 0\,, \tag{33}$$

which is of class 2 whatever the choice of the orderly ranking. Note that here and in subsequent sections, where Janet-Riquier theory is applied, the *symbol variables* V_{abcd}, V_{abcde} have no connection to Vessiot vector fields V. This yields

$$\alpha_1^{(1)} = 2, \quad \alpha_1^{(2)} = 1\,.$$

Trivially, all variables are multiplicative variables and the symbol of one equation is always involutive. But, we must examine whether this system is formally integrable. We only need to compute $\mathcal{R}_2$ and its projection $\mathcal{R}_1^{(1)}$ and check whether $\mathcal{R}_2 = \mathcal{R}_1^{(1)}$, where $\dim(\mathcal{R}_1) = 5$ and $\dim(\mathcal{M}_1) = 3$. Our prolonged system of equations $\mathcal{R}_2$ consists of $f^{(R)}_{1212}, f^{(R)}_{1212,1}, f^{(R)}_{1212,2}$, namely,

$$\begin{aligned}
0 &= R_{1212} - 2P_{1212} + 2P_{1221} + 2L_{121}(\Gamma^1_{12} - \Gamma^2_{22}) - 2L_{122}(\Gamma^1_{11} + \Gamma^2_{12})\\
0 &= R_{1212,1} - 2S_{12121} + 2S_{12211} + 2P_{1211}(\Gamma^1_{12} - \Gamma^2_{22})\\
&\quad -2P_{1221}(\Gamma^1_{11} + \Gamma^2_{12}) + 2L_{121}(\Gamma^1_{12} - \Gamma^2_{22})_{,1} - 2L_{122}(\Gamma^1_{11} + \Gamma^2_{12})_{,1}\\
0 &= R_{1212,2} - 2S_{12122} + 2S_{12212} + 2P_{1212}(\Gamma^1_{12} - \Gamma^2_{22})\\
&\quad -2P_{1222}(\Gamma^1_{11} + \Gamma^2_{12}) + 2L_{121}(\Gamma^1_{12} - \Gamma^2_{22})_{,2} - 2L_{122}(\Gamma^1_{11} + \Gamma^2_{12})_{,2}\,.
\end{aligned} \tag{34}$$

No integrability conditions can be created because

$$\dim(\mathcal{R}_1^{(1)}) = \dim(\mathcal{R}_2) - \dim(\mathcal{M}_2) = 9 - 4 = 5 = \dim(\mathcal{R}_1)$$

and the system is linear so that formal integrability has been shown and the system $\mathcal{R}_1$ is involutive.

If we incorporate the differential gauge condition $L_{ab}{}^{s}{}_{;s} = 0$, then $\mathcal{M}_1$ is formed by

$$\begin{aligned}
-2V_{1212} + 2V_{1221} &= 0\\
g^{11}V_{1211} + g^{12}V_{1221} + g^{12}V_{1212} + g^{22}V_{1222} &= 0\,,
\end{aligned} \tag{35}$$

which has two equations of class 2 again whatever the choice of the orderly ranking be and we obtain $\alpha_1^{(1)} = 2$, $\alpha_1^{(2)} = 0$. In order to check whether $\mathcal{M}_1$ is involutive, we prolong to the corresponding $\mathcal{M}_2$ which is formed by

$$\begin{aligned}
-2V_{12112} + 2V_{12211} &= 0\\
-2V_{12122} + 2V_{12212} &= 0\\
g^{11}V_{12111} + g^{12}V_{12211} + g^{12}V_{12112} + g^{22}V_{12212} &= 0\\
g^{11}V_{12112} + g^{12}V_{12212} + g^{12}V_{12122} + g^{22}V_{12222} &= 0\,,
\end{aligned} \tag{36}$$

We get $r(\mathcal{M}_2) = 4 = k \cdot \beta_1^{(k)}$ and therefore $\mathcal{M}_1$ is involutive. We must check that the new system $\mathcal{R}_1$ is formally integrable again and so add the following 3 equations $L_{12}{}^{s}{}_{;s}$, $\partial_1 L_{12}{}^{s}{}_{;s}$, $\partial_2 L_{12}{}^{s}{}_{;s}$ to (34):

$$\begin{aligned}
0 &= g^{11}P_{1211} + g^{12}(P_{1221} + P_{1212}) + g^{22}P_{1222} - L_{121}\delta_3 - L_{122}\delta_4\\
0 &= g^{11}S_{12111} + g^{12}(S_{12211} + S_{12112}) + g^{22}S_{12212} + g^{11}_{,1}P_{1211} + g^{12}_{,1}(P_{1221} + P_{1212})\\
&\quad + g^{22}_{,1}P_{1222} - P_{1211}\delta_3 - P_{1221}\delta_4 - L_{121}\delta_{3,1} - L_{122}\delta_{4,1}\\
0 &= g^{11}S_{12112} + g^{12}(S_{12212} + S_{12122}) + g^{22}S_{12222} + g^{11}_{,2}P_{1211} + g^{12}_{,2}(P_{1221} + P_{1212})\\
&\quad + g^{22}_{,2}P_{1222} - P_{1212}\delta_3 - P_{1222}\delta_4 - L_{121}\delta_{3,2} - L_{122}\delta_{4,2}\,,
\end{aligned} \tag{37}$$

where δ_3, δ_4 are as defined in Appendix A. The system (34) together with (37) now forms the prolonged system of PDEs $\mathcal{R}_2$ including the differential gauge condition and its partial derivatives. Again, we cannot solve any of the 4 second-order PDEs in such a way that no second-order derivatives remain. In order to show this, we can use the symbol matrix of $\mathcal{M}_2$ as well:

V_{12111}	V_{12112}	V_{12122}	V_{12211}	V_{12212}	V_{12222}
0	-2	0	2	0	0
0	0	-2	0	2	0
g^{11}	g^{12}	0	g^{12}	g^{22}	0
0	g^{11}	g^{12}	0	g^{12}	g^{22}

,

which has maximal rank $r(\mathcal{M}_2) = 4$ and no integrability conditions can arise because of the linearity of the system. This means that $\mathcal{R}_1^{(1)} = \mathcal{R}_1$, thus making $\mathcal{R}_1$ formally integrable and involutive. We conclude:

Theorem 6 *Proposition*
In any two-dimensional spacetime, the Riemann-Lanczos equations are in involution with respect to the spacetime coordinates and the differential gauge condition does not affect the result. The corresponding Cartan characters (s_0, s_1, s_2) *for the involutory systems are (3,2,1) when no differential gauge condition is imposed but (4,2,0) when the differential gauge condition is imposed.*

C. General Solution to the Riemann-Lanczos Problem in 2 Dimensions and Characteristic Coordinates

It is instructive to repeat some of the above calculations using different local coordinates. First, we look at the general 2-dimensional spacetime with Lorentzian signature and line element in characteristic coordinates x^1, x^2 written as

$$ds^2 = -e^{2\rho} dx^1 dx^2 \ ,$$

where ρ is an arbitrary function of x^1, x^2. The only two equations we have to solve are $f^{(R)}_{1212} = 0, L_{12;s}^{\ \ \ s} = 0$:

$$\begin{aligned} 0 &= -e^{2\rho}\rho_{,x^1x^2} + 4L_{121}\rho_{,x^2} - 4L_{122}\rho_{,x^1} - 2P_{1212} + 2P_{1221} \ , \\ 0 &= 2e^{-2\rho}(2L_{121}\rho_{,x^2} + 2L_{122}\rho_{,x^1} - P_{1212} - P_{1221}) \ . \end{aligned} \tag{38}$$

We find that the general solution is of the form

$$\begin{aligned} L_{121} &= e^{2\rho} f_1(x^1) - \frac{1}{4} e^{2\rho}\rho_{,x^1} \ , \\ L_{122} &= e^{2\rho} f_2(x^2) + \frac{1}{4} e^{2\rho}\rho_{,x^2} \ , \end{aligned} \tag{39}$$

where $f_1(x^1), f_2(x^2)$ are 2 arbitrary functions depending on one local coordinate each. This is in agreement with the result for the Cartan characters claiming that the general solution depends on two arbitrary functions of one variable each. But can we obtain the intrinsic values for the characters using the above coordinate frame? The symbol derived from (38) is

$$\begin{aligned} 0 &= -2V_{1212} + 2V_{1221} \\ 0 &= -V_{1212} - V_{1221} \ , \end{aligned} \tag{40}$$

from which we obtain $\beta_1^{(1)} = \beta_1^{(2)} = 1$ based on the ranking stemming from $x^2 \succ x^1$. This leads to $\alpha_1^{(1)} = \alpha_1^{(2)} = 1$, which **cannot be** the intrinsic result as we already solved the 2-dimensional problem in the previous section. We see that in (40) only one variable of class 2, namely V_{1212}, and only one variable of class 1, namely V_{1221}, occurs whereas neither V_{1211} nor V_{1222} occur at all to make both equations be of class 2. As this can happen in another coordinate system as we shall see for the 3-dimensional case below as well, the above coordinate frame is not δ-regular and we need to perform a coordinate transformation of the form

$$\begin{aligned} d\tilde{x}^1 &= a_{11}dx^1 + a_{12}dx^2 \\ d\tilde{x}^2 &= a_{21}dx^1 + a_{22}dx^2 \end{aligned} \tag{41}$$

in order to obtain the correct values for $\alpha_1^{(1)}, \alpha_1^{(2)}$. After such a transformation, the new symbol in orthonomic form is

$$\begin{aligned} V_{1222} &= -\frac{a_{21}}{a_{22}} V_{1221} \\ V_{1212} &= -\frac{a_{11}}{a_{12}} V_{1211} \, . \end{aligned} \tag{42}$$

Now, both equations are of class 2 so that $\beta_1^{(1)} = 0, \beta_1^{(2)} = 2$ which produces $\alpha_1^{(1)} = 2, \alpha_1^{(2)} = 0$ which is the known intrinsic result for the Cartan characters.

For spaces with Euclidean signature we can write their line element as

$$ds^2 = e^{2\rho}((dx^1)^2 + (dx^2)^2) \, ,$$

where ρ is again an arbitrary function of x^1, x^2 leading to the system of equations

$$\begin{aligned} 0 &= -e^{2\rho}(\rho_{,x^1x^1} + \rho_{,x^2x^2}) + 4L_{121}\rho_{,x^2} - 4L_{122}\rho_{,x^1} - 2P_{1212} + 2P_{1221} \, , \\ 0 &= e^{-2\rho}(-2L_{121}\rho_{,x^1} - 2L_{122}\rho_{,x^2} + P_{1211} + P_{1222}) \, . \end{aligned} \tag{43}$$

The solution to (43) resembles the one for (38) and is

$$\begin{aligned} L_{121} &= e^{2\rho}(f_1(x^1 + x^2) - \frac{1}{2}\rho_{,x^2}) \\ L_{122} &= e^{2\rho}(f_2(x^1 - x^2) + \frac{1}{2}\rho_{,x^1}) \, . \end{aligned} \tag{44}$$

In this case we find the intrinsic values for the characters to be $\alpha_1^{(1)} = 2$, $\alpha_1^{(2)} = 0$ using (43) directly due to the fact that here we did not use characteristic local coordinates.

IV. The Riemann-Lanczos Problem in 3 Dimensions: Involution or Prolongation?

In this section we shall discuss the Riemann-Lanczos problem in 3 dimensions as an EDS and as a Janet-Riquier system of partial differential equations (PDEs). First, we shall use EDS theory to find out more about the problem.

A. The 3-dimensional Riemann-Lanczos Problem as an EDS

In 3 spacetime dimensions with local coordinates x^1, x^2, x^3, we obtain 8 independent components of the Lanczos tensor, namely, $L_{121}, L_{122}, L_{131}, L_{133}$,

$L_{232}, L_{233}, L_{123}, L_{132}$ when imposing the cyclic conditions (1). Each of them has three first-order partial derivatives so that we use the jet bundle $\mathcal{J}^1(\mathbb{R}^3, \mathbb{R}^8)$ with formal dimension $N = 35$ to express the EDS. We obtain the 6 independent Riemann-Lanczos equations whose exterior derivatives are $df^{(R)}_{1212}, df^{(R)}_{1313}, df^{(R)}_{2323}, df^{(R)}_{1213}, df^{(R)}_{1223}, df^{(R)}_{1323}$. These latter equations can locally be written as

$$\begin{aligned}
0 = &-2dP_{1212} + 2dP_{1221} + dL_{121}(2\Gamma^1_{12} + 2\Gamma^2_{22}) - dL_{122}(2\Gamma^1_{11} + 2\Gamma^2_{12}) + 2\Gamma^3_{22} \\
&\cdot dL_{131} + 2\Gamma^3_{11} dL_{232} + 2\Gamma^3_{12} dL_{123} - 4\Gamma^3_{12} dL_{132} + (R_{1212,e} + \alpha_{1212e})dx^e \\
0 = &-2dP_{1313} + 2dP_{1331} + dL_{131}(2\Gamma^1_{13} + 2\Gamma^3_{33}) - dL_{133}(2\Gamma^1_{11} + \Gamma^3_{13}) + 2\Gamma^2_{33} \\
&\cdot dL_{121} - 2\Gamma^2_{11} dL_{233} - 4\Gamma^2_{13} dL_{123} + 2\Gamma^2_{13} dL_{132} + (R_{1313,e} + \alpha_{1313e})dx^e \\
0 = &-2dP_{2323} + 2dP_{2332} + 2dL_{232}(\Gamma^2_{23} + \Gamma^3_{33}) - 2dL_{233}(\Gamma^2_{22} + \Gamma^3_{23}) - 2\Gamma^1_{33} \\
&\cdot dL_{122} - 2\Gamma^1_{22} dL_{133} + 2\Gamma^1_{23} dL_{123} + 2\Gamma^1_{23} dL_{132} + (R_{2323,e} + \alpha_{2323e})dx^e \\
0 = &-dP_{1213} + dP_{1231} - dP_{1312} + dP_{1321} + dL_{121}(\Gamma^1_{13} + 2\Gamma^2_{23}) + dL_{131} \\
&\cdot(\Gamma^1_{12} + 2\Gamma^3_{23}) - \Gamma^3_{12} dL_{133} - \Gamma^2_{13} dL_{122} - \Gamma^2_{11} dL_{232} + \Gamma^3_{11} dL_{233} - dL_{123} \\
&\cdot(\Gamma^1_{11} + 2\Gamma^2_{12} - \Gamma^3_{13}) - dL_{132}(2\Gamma^3_{13} + \Gamma^1_{11} - \Gamma^2_{12}) + (R_{1213,e} + \alpha_{1213e})dx^e \\
0 = &-dP_{1223} + 2dP_{1232} - dP_{1322} + dP_{2321} + dL_{122}(2\Gamma^1_{13} + \Gamma^2_{23}) - dL_{232} \\
&\cdot(\Gamma^2_{12} + 2\Gamma^3_{13}) - \Gamma^1_{23} dL_{121} + \Gamma^3_{12} dL_{233} + \Gamma^1_{22} L_{131} - \Gamma^3_{22} dL_{133} - dL_{123} \\
&\cdot(\Gamma^1_{12} + 2\Gamma^2_{22} + \Gamma^3_{23}) + dL_{132}(2\Gamma^3_{23} + \Gamma^2_{22} - \Gamma^1_{12}) + (R_{1223,e} + \alpha_{1223e})dx^e \\
0 = &-2dP_{1323} + dP_{1332} + dP_{1233} + dP_{2331} - dL_{133}(2\Gamma^1_{12} + \Gamma^3_{23}) - dL_{233} \\
&\cdot(2\Gamma^2_{12} + \Gamma^3_{13}) + \Gamma^2_{13} dL_{232} + \Gamma^2_{33} dL_{122} + \Gamma^1_{23} L_{131} - \Gamma^1_{33} dL_{121} - dL_{123}(2\Gamma^2_{23} \\
&-\Gamma^3_{33} - \Gamma^1_{13}) + dL_{132}(\Gamma^1_{13} + 2\Gamma^3_{33} + \Gamma^2_{23}) + (R_{1323,e} + \alpha_{1323e})dx^e \,, \qquad (45)
\end{aligned}$$

where α_{abcde} is defined in Appendix B. The polar matrix consists of 35 columns for the $(dx^e, dL_{abc}, dP_{abcd})$ and we split the first 6 rows stemming from (45) into 3 blocks which we call $(M_{01} \quad M_{02} \quad M_{03})$ so that the full polar matrix of $H(E^3)_x$ is given by

$$\begin{pmatrix} M_{01} & M_{02} & M_{03} \\ M(P) & 1 & 0 \\ M(U) & 0 & N(U) \\ M(V) & 0 & N(V) \\ M(Z) & 0 & N(Z) \end{pmatrix} ,$$

where we denote the truncated matrix obtained by leaving out the rows $(M_{01}\; M_{02}\; M_{03})$ by M_1. All the matrix expressions can be found in Appendix B and a more detailed explanation is given in [**?**]. The first 8 rows from M_1 together with $(M_{01}\; M_{02}\; M_{03})$ lead to $s_0 = s'_0 = 14$. Then, for s_1 and s'_1 there is neither a linear combination of the full polar system nor one of the reduced polar system possible. We have $s_1 = s'_1 = 8$ because the rank of the relevant 22x35-matrix consisting of $(M_{01} \quad M_{02} \quad M_{03})$ and the first 16 rows of M_1 is $22 = s_0 + s_1 = s'_0 + s'_1$ and so $s_0 = s'_0 = 8$. But now, for the second reduced polar system, we find that there is exactly one linear combination possible which we can write down as

$$\begin{aligned}
0 = &A_1 \cdot (row_1) + A_2 \cdot (row_2) + \cdots + A_6 \cdot (row_6) \\
&B_1 \cdot (row_7) + B_2 \cdot (row_8) + \cdots + B_8 \cdot (row_{14}) \\
&+C_1 \cdot (row_{15}) + C_2 \cdot (row_{16}) + \cdots + C_8 \cdot (row_{22}) \,, \qquad (46)
\end{aligned}$$

where the linear multipliers are given in Appendix B. If we wish that this linear combination for the full polar system be valid, we have to claim that the 3 equations (for $e = 1, 2, 3$) hold

$$
\begin{aligned}
0 = &\tilde{\alpha}_{1212e} A_1 + \tilde{\alpha}_{1313e} A_2 + \tilde{\alpha}_{2323e} A_3 + \tilde{\alpha}_{1213e} A_4 + \tilde{\alpha}_{1223e} A_5 + \tilde{\alpha}_{1323e} A_6 \\
&- U_{121e} B_1 - U_{122e} B_2 - U_{131e} B_3 - U_{133e} B_4 - U_{232e} B_5 - U_{233e} B_6 \\
&- U_{123e} B_7 - U_{132e} B_8 - V_{121e} C_1 - V_{122e} C_2 - V_{131e} C_3 - V_{133e} C_4 \\
&- V_{232e} C_5 - V_{233e} C_6 - V_{123e} C_7 - V_{132e} C_8 \,,
\end{aligned} \tag{47}
$$

where $\tilde{\alpha}_{abcde}$ is as defined in Appendix B, without imposing any further restrictions on the first two Vessiot vector fields U, V. This can easily be checked by looking at how many of the U_{abcd} and how many of the V_{abcd} can be chosen arbitrarily. For the 24 U_{abcd} there are 6 restrictions, so 18 can be chosen arbitrarily. For the V_{abcd} we have 14 restrictions, so 10 remain arbitrary. Based on this argument, we find that some of the U_{abcd}, V_{abcd} never cancel out and the linear combination (46) does not hold for the full system so that $s_2 = s'_2 + 1$ such that $s'_2 = 7$ and $s_2 = 8$ [**?**]. This system is clearly not in involution and its reduced characters (s'_0, s'_1, s'_2, s'_3) are $(14, 8, 7, 3)$. We can test this result with the REDUCE code given in [**?**] and immediately obtain (8,7,3) for the reduced characters (s'_1, s'_2, s'_3) and that the system is not in involution. In order to carry out this computation it is sufficient to use an arbitrary diagonalized line element which describes 3 -dimensional spacetimes completely as shown in [**?**].

We can also use the tableau matrix to calculate this result as shown next. First, we must complete our set of 14 one-forms together with the 3 ω^e to a complete coframe consisting of $N = 35$ elements by introducing the 18 π^Λ such that $(df^{(R)}_{abcd}, K_{abc}, \omega^e, \pi^\Lambda)$ forms a cobasis on $\mathcal{M}$. We introduce the 18 π^Λ using collective indices Λ with $\Lambda = 1, \cdots, 18$ with the following correspondence between $\pi^\Lambda \leftrightarrow dP_{abcd}$:

$$
\begin{array}{lll}
\pi^1 \leftrightarrow dP_{1211} \,, & \pi^2 \leftrightarrow dP_{1221} \,, & \pi^3 \leftrightarrow dP_{1311} \,, \\
\pi^4 \leftrightarrow dP_{1331} \,, & \pi^5 \leftrightarrow dP_{2321} \,, & \pi^6 \leftrightarrow dP_{2331} \,, \\
\pi^7 \leftrightarrow dP_{1231} \,, & \pi^8 \leftrightarrow dP_{1321} \,, & \pi^9 \leftrightarrow dP_{1222} \,, \\
\pi^{10} \leftrightarrow dP_{1312} \,, & \pi^{11} \leftrightarrow dP_{1332} \,, & \pi^{12} \leftrightarrow dP_{2322} \,, \\
\pi^{13} \leftrightarrow dP_{2332} \,, & \pi^{14} \leftrightarrow dP_{1232} \,, & \pi^{15} \leftrightarrow dP_{1322} \,, \\
\pi^{16} \leftrightarrow dP_{1333} \,, & \pi^{17} \leftrightarrow dP_{2333} \,, & \pi^{18} \leftrightarrow dP_{1233} \,.
\end{array}
$$

When using the correspondence $\theta^\alpha \leftrightarrow K_{abc}$ based on equations (18) in [**?**] the 8 dK_{abc} can be recast as

$$
\begin{aligned}
d\theta^1 &\equiv -\pi^1 \wedge \omega^1 - \pi^2 \wedge \omega^2 - \pi^7 \wedge \omega^3 + \pi^{10} \wedge \omega^3 - \pi^8 \wedge \omega^3 \\
&\quad + \frac{1}{2} B^1_{i2} \omega^i \wedge \omega^2 + \frac{1}{2} B^1_{i3} \omega^i \wedge \omega^3 \\
d\theta^2 &\equiv -\pi^2 \wedge \omega^1 - \pi^9 \wedge \omega^2 - 2\pi^{14} \wedge \omega^3 - \pi^5 \wedge \omega^3 + \frac{1}{2} B^2_{i3} \omega^i \wedge \omega^3 \\
d\theta^3 &\equiv -\pi^3 \wedge \omega^1 - \pi^{10} \wedge \omega^2 - \pi^4 \wedge \omega^3 + \frac{1}{2} B^3_{i3} \omega^i \wedge \omega^3 \\
d\theta^4 &\equiv -\pi^4 \wedge \omega^1 - \pi^{11} \wedge \omega^2 - \pi^{16} \wedge \omega^3 \\
d\theta^5 &\equiv -\pi^5 \wedge \omega^1 - \pi^{12} \wedge \omega^2 - \pi^{13} \wedge \omega^3 + \frac{1}{2} B^5_{i3} \omega^i \wedge \omega^3 \\
d\theta^6 &\equiv -\pi^6 \wedge \omega^1 - \pi^{13} \wedge \omega^2 - \pi^{17} \wedge \omega^3 \\
d\theta^7 &\equiv -\pi^7 \wedge \omega^1 - \pi^{14} \wedge \omega^2 - \pi^{18} \wedge \omega^3 \\
d\theta^8 &\equiv -\pi^8 \wedge \omega^1 - \pi^{15} \wedge \omega^2 - \frac{1}{2}\pi^{11} \wedge \omega^3 - \frac{1}{2}\pi^8 \wedge \omega^3 - \frac{1}{2}\pi^6 \wedge \omega^3 \\
&\quad + \frac{1}{2} B^8_{i3} \omega^i \wedge \omega^3 \,.
\end{aligned} \tag{48}
$$

We can easily calculate the reduced characters when using the tableau matrices derived from (48) and obtain $s'_1 = 8$, $s'_2 = 7$ and $s'_3 = 3$. We also find that all remaining torsion coefficients B^{α}_{ij} in (48) can be absorbed so that no integrability conditions can occur for the system (45) itself. However, because we have the rank deficiency in the reduced polar matrix leading to $s_2 = s'_2 + 1$, we must prolongate the system to second order. We introduce new jet variables S_{abcde}, of which the projection back onto the spacetime manifold corresponds to $S_{abcde} := P_{abcd,e} = L_{abc,de}$, and denote the 24 contact conditions arising from them by K_{abcd}. The prolonged Pfaffian system then is given by

$$\begin{aligned} 0 &= f^{(R)}_{abcd} \\ 0 &= f^{(R)}_{abcd,e} \\ 0 &= df^{(R)}_{abcd} \\ 0 &= df^{(R)}_{abcd,e} \\ K_{abc} &= dL_{abc} - P_{abcd}dx^e \\ K_{abcd} &= dP_{abcd} - S_{abcde}dx^e \end{aligned} \tag{49}$$

together with the exterior derivatives dK_{abc} and dK_{abcd}. Calculations using an adapted REDUCE computer code suggest that the system (49) is not in involution and integrability conditions occur.

Note that instead of the prolongation to (49) we can carry out the same prolongation which was carried out for the Riemann-Lanczos problem in 4 dimensions in [**?**]. This may lead an involutory system eventually but we leave this as future work. However, the process of prolongation often becomes simpler when Janet-Riquier theory is used what we are going to next.

B. The 3-dimensional Riemann-Lanczos Problem as a System of PDEs

The symbol $\mathcal{M}_1$ for the unprolonged Riemann-Lanczos problem in 3 dimensions consists of the 6 linear equations derived from $f^{(R)}_{1212}, f^{(R)}_{1313}, f^{(R)}_{2323}, f^{(R)}_{1213}$, $f^{(R)}_{1223}, f^{(R)}_{1323}$ and is given in orthonomic form as

$$\begin{array}{ll} V_{1212} = V_{1221} & x^1\ x^2\ \bullet \\ V_{1313} = V_{1331} & x^1\ x^2\ x^3 \\ V_{2323} = V_{2332} & x^1\ x^2\ x^3 \\ V_{1213} = V_{1231} - V_{1312} + V_{1321} & x^1\ x^2\ x^3 \\ V_{1223} = 2V_{1232} - V_{1322} + V_{2321} & x^1\ x^2\ x^3 \\ V_{1323} = \frac{1}{2}(V_{1332} + V_{1233} + V_{2331}) & x^1\ x^2\ x^3. \end{array}$$

We imposed the ranking $x^3 \succ x^2 \succ x^1$ which then induced the ranking $P_{abc3} \succ P_{abc2} \succ P_{abc1}$ and amongst each set P_{abce} the ranking $P_{233e} \succ P_{232e} \succ P_{132e} \succ P_{123e} \succ P_{133e} \succ P_{131e} \succ P_{122e} \succ P_{121e}$. From this we now obtain $\beta^{(1)}_1 = 0, \beta^{(2)}_1 = 1, \beta^{(3)}_1 = 5$ and $\alpha^{(1)}_1 = 8, \alpha^{(2)}_1 = 7, \alpha^{(3)}_1 = 3$. But, we must check whether we actually used δ-regular coordinates and we perform the following coordinate transformation

$$\begin{aligned} d\tilde{x}^1 &= a_{11}dx^1 + a_{12}dx^2 + a_{13}dx^3 \\ d\tilde{x}^2 &= a_{21}dx^1 + a_{22}dx^2 + a_{23}dx^3 \\ d\tilde{x}^3 &= a_{31}dx^1 + a_{32}dx^2 + a_{33}dx^3 \,, \end{aligned} \tag{50}$$

which leaves us with the symbol in orthonomic form in the new coordinates

$$\begin{aligned} V_{1223} &= \frac{1}{a_{13}}(B_1 + a_{23}V_{1213}) \\ V_{1333} &= \frac{1}{a_{13}}(B_2 + a_{33}V_{1313}) \\ V_{2333} &= \frac{1}{a_{23}}(B_3 + \frac{a_{33}}{a_{13}}(B_5 + \frac{a_{33}}{a_{13}}B_1 + \frac{a_{23}}{a_{13}}B_4 \\ &\quad +2V_{1213}\frac{a_{33}a_{23}}{a_{13}} - 3a_{23}V_{1233} + \frac{a_{23}^2}{a_{13}}V_{1313}) \\ V_{1323} &= \frac{1}{a_{13}}(B_4 + a_{33}V_{1213} - a_{13}V_{1233} + a_{23}V_{1313}) \\ V_{2332} &= f(V_{abc1}, V_{abc2}) \,. \end{aligned} \tag{51}$$

In (51) we dropped the   and the B_i and $f(V_{abc1}, V_{abc2})$ are given in Appendix C. We see from (51) and Appendix C that no variable of the form V_{abc3} remains in the last equation for any choice of the a_{ij} in (50) so that we obtain 5 equations of class 3 and one of class 2. This means that our original coordinates were δ-regular and we can use our first coordinate system in order to obtain intrinsic results.

In order to see whether the symbol is involutive, we differentiate each equation with respect to x^1, x^2, x^3 from which we can produce $\mathcal{M}_2$, where its sparse coefficient matrix is given and explained in detail in [**?**]. We can determine the rank of this sparse matrix easily by hand (or using Maple) and obtain $r(\mathcal{M}_2) = 18$ as shown in [**?**]. We now compute the total number of multiplicative variables

$$\sum_{k=1}^{3} k \cdot \beta_1^{(k)} = 17 \neq r(\mathcal{M}_2) = 18 \,,$$

which means that even the necessary condition for the symbol to be involutive is not fulfilled and the system cannot be in involution.

Because $\mathcal{M}_1$ is not involutive, we prolong $\mathcal{R}_1$ to $\mathcal{R}_2$ which consists of 24 equations 18 of which are the partial derivatives of the 6 $f_{abcd}^{(R)}$. We write the prolonged symbol $\mathcal{M}_2$ down which is based on the following ranking. We order the S_{abcde} such that $S_{abc33} \succ S_{abc23} \succ S_{abc22} \succ S_{abc13} \succ S_{abc12} \succ S_{abc11}$ and then amongst each set $S_{233ij} \succ S_{232ij} \succ S_{132ij} \succ S_{123ij} \succ S_{133ij} \succ S_{131ij} \succ S_{122ij} \succ S_{121ij}$. We find that the symbol in orthonomic form is given by (with multiplicative variables indicated in each equation)

$$
\begin{array}{llll}
\boxed{1} & V_{12112} & = V_{12211} & x^1 \bullet \bullet \\
\boxed{2} & V_{12122} & = V_{12212} & x^1\, x^2 \bullet \\
\boxed{3} & V_{12123} & = 2V_{12312} - V_{13212} + V_{23211} & x^1\, x^2 \bullet \\
& & & \\
\boxed{4} & V_{13113} & = V_{13311} & x^1 \bullet \bullet \\
\boxed{5} & V_{13123} & = V_{13312} & x^1\, x^2 \bullet \\
\boxed{6} & V_{13133} & = V_{13313} & x^1\, x^2\, x^3 \\
& & & \\
\boxed{7} & V_{23213} & = V_{23312} & x^1 \bullet \bullet \\
\boxed{8} & V_{23223} & = V_{23322} & x^1\, x^2 \bullet \\
\boxed{9} & V_{23233} & = V_{23323} & x^1\, x^2\, x^3 \\
& & & \\
\boxed{10} & V_{12113} & = V_{12311} - V_{13112} + V_{13211} & x^1 \bullet \bullet \\
\boxed{11} & V_{13122} & = 2V_{13212} - V_{12312} - V_{23211} & x^1\, x^2 \bullet \\
\boxed{12} & V_{12133} & = \frac{3}{2}V_{12313} - \frac{1}{2}V_{13312} + \frac{1}{2}V_{23311} & x^1\, x^2\, x^3 \\
& & & \\
\boxed{13} & V_{12213} & = 2V_{12312} - V_{13212} + V_{23211} & x^1 \bullet \bullet \\
\boxed{14} & V_{12223} & = 2V_{12322} - V_{13222} + V_{23212} & x^1\, x^2 \bullet \\
\boxed{15} & V_{12233} & = \frac{3}{2}V_{12323} + \frac{1}{2}V_{23312} - \frac{1}{2}V_{13322} & x^1\, x^2\, x^3 \\
& & & \\
\boxed{16} & V_{13213} & = \frac{1}{2}(V_{13312} + V_{12313} + V_{23311}) & x^1 \bullet \bullet \\
\boxed{17} & V_{13223} & = \frac{1}{2}(V_{13322} + V_{12323} + V_{23312}) & x^1\, x^2 \bullet \\
\boxed{18} & V_{13233} & = \frac{1}{2}(V_{13323} + V_{12333} + V_{23313}) & x^1\, x^2\, x^3 .
\end{array}
$$

This system produces 6 equations of class 1, 7 of class 2 and 5 of class 3 leading to $\beta_2^{(1)} = 6, \beta_2^{(2)} = 7, \beta_2^{(3)} = 5$. The total number of multiplicative variables equals 35 and if $\mathcal{M}_2$ were involutive, we would have to obtain $r(\mathcal{M}_3) = 35$. Differentiating $\mathcal{R}_2$ with respect to all 3 spacetime coordinates, we obtain 54 equations relevant for the symbol $\mathcal{M}_3$. But all those equations obtained from differentiation with respect to non-multiplicative variables are superfluous as each of them can be obtained by differentiating with respect to multiplicative variables except for $\partial_3 f^{(R)}_{1212}$. Therefore, $r(\mathcal{M}_3) \leq 36$, when we use formal differentiation of the above symbol equations directly because $\partial_3(eqn.3)$ corresponding to $\partial_3 f^{(R)}_{1212}$ is a linear combination of some other equations in $\mathcal{M}_3$. We obtain $r(\mathcal{M}_3) = 35$, which means that $\mathcal{M}_2$ is involutive.

We obtain that the symbol equation $\partial_3(eqn.3)$ can also be created as a linear combination of

$$
\partial_3(eqn.3) = 2\partial_2(eqn.16) + 2\partial_3(eqn.11) - 2\partial_3(eqn.13) - \partial_2(eqn.5) - \partial_1(eqn.7) \tag{52}
$$

using formal differentiation so that $r(\mathcal{M}_3) = \sum_{k=1}^{3} k \cdot \beta_2^{(k)} = 35$. However, it turns out that when (52) is rewritten in terms of the full equations from $\mathcal{R}_3$ as

$$
I = f^{(R)}_{1212,33} + f^{(R)}_{1313,22} + f^{(R)}_{2323,11} - 2f^{(R)}_{1323,12} - 2f^{(R)}_{1213,23} + 2f^{(R)}_{1223,13}\,, \tag{53}
$$

it is not a trivial identity any longer and cannot be obtained by means of any linear combination of the $f^{(R)}_{abcd}$ and their derivatives $f^{(R)}_{abcd,e}$ so that

$r(\mathcal{R}_3) = 6 + 18 + 36 = 60$. Because our system is linear, it is instructive to compute

$$\begin{aligned} \dim(\mathcal{R}_1) &= 18\,, & \dim(\mathcal{M}_1) &= 18\,, \\ \dim(\mathcal{R}_2) &= 56\,, & \dim(\mathcal{M}_2) &= 30\,, \\ \dim(\mathcal{R}_3) &= 100\,, & \dim(\mathcal{M}_3) &= 45\,, \end{aligned}$$

which leads us to the important consequence that

$$\dim(\mathcal{R}_2^{(1)}) = \dim(\mathcal{R}_3) - \dim(\mathcal{M}_3) = 100 - 45 = 55 = \dim(\mathcal{R}_2) - 1\,.$$

But this means that **one integrability condition of the form (53) occurs**. Our new system of equations has to be given by

$$\begin{aligned} 0 &= f^{(R)}_{abcd} \\ 0 &= f^{(R)}_{abcd,e} \\ 0 &= I\,. \end{aligned} \tag{54}$$

We find that its symbol is simply given by the previous $\mathcal{M}_2$ together with

$$V_{12323} = \frac{1}{\Gamma^3_{12} - \Gamma^3_{13}}[V_{23312}(\Gamma^3_{13} - \Gamma^3_{12}) + 2\Gamma^3_{33}V_{12312} - V_{13322}(2\Gamma^1_{11} + \Gamma^3_{12} + \Gamma^3_{13})] \tag{55}$$

which adopts 3 multiplicative variables. Therefore:

$$\beta^{(1)}_{2,(1)} = 6, \beta^{(2)}_{2,(1)} = 7, \beta^{(3)}_{2,(1)} = 6 \text{ so that } \alpha^{(1)}_{2,(1)} = 18, \alpha^{(2)}_{2,(1)} = 9, \alpha^{(3)}_{2,(1)} = 2\,.$$

We see that $r(\mathcal{M}_3^{(1)}) = r(\mathcal{M}_3) + 3 = 38$ because the 3 derivatives resulting from (55) are linearly independent and because

$$r(\mathcal{R}_3^{(1)}) = r(\mathcal{R}_1) + r(\mathcal{R}_2^{(1)}) + 35 + 3 = 63$$

which leads to

$$\dim(\mathcal{R}_2^{(2)}) = \dim(\mathcal{R}_3^{(1)}) - \dim(\mathcal{M}_3^{(1)}) = 97 - 42 = 55 = \dim(\mathcal{R}_2^{(1)})\,.$$

Because $r(\mathcal{M}_3^{(1)}) = \sum_{k=1}^{3} k \cdot \beta^{(k)}_{2,(1)} = 6 + 14 + 18 = 38$, $\mathcal{M}_2^{(1)}$ is involutive and because $\dim(\mathcal{R}_2^{(2)}) = \dim(\mathcal{R}_2^{(1)}) = 55$, there are no integrability conditions that can occur and so the system $\mathcal{R}_2^{(1)}$ is in involution.

C. A Covariant Formulation of the Integrability Condition for $n = 3$

Even though the above prolongation of the Riemann-Lanczos problem mathematically leads to a prolonged system in involution, this prolonged system (54) does not respect general covariance. This is not satisfactory from a general relativity point of view and we are going to suggest a covariant version of the above prolongation. Instead of adding the partial derivatives to our equations, we look at the following new system of equations

$$\begin{aligned} 0 &= f^{(R)}_{abcd} \\ 0 &= f^{(R)}_{abcd;e}\,. \end{aligned} \tag{56}$$

The symbol of this modified system obviously coincides with that of the system formed by $f^{(R)}_{abcd}, f^{(R)}_{abcd,e}$ and again, we obtain for the rank of $\mathcal{M}_3$ that

$r(\mathcal{M}_3) = \sum_{k=1}^{3} k \cdot \beta_2^k = 35$ so that $\mathcal{M}_2$ is involutive. We now have to see whether the linear combination (52) also holds for the full covariant system here. Again, calculations show that this is not the case and we obtain a covariant version of the above integrability condition in solved form given by

$$\begin{aligned} I_{Cov} = f^{(R)}_{1212;33} + f^{(R)}_{1313;22} + f^{(R)}_{2323;11} - f^{(R)}_{1323;12} - f^{(R)}_{1323;21} - f^{(R)}_{1213;23} - f^{(R)}_{1213;32} \\ +f^{(R)}_{1223;13} + f^{(R)}_{1223;31} \; . \end{aligned} \tag{57}$$

This condition can be rewritten again in solved form as

$$I_{Cov} = B_{12123;3} + B_{13132;2} + B_{23231;1} \; , \tag{58}$$

where $B_{abcde} := f^{(R)}_{ab[cd;e]}$ which amounts to the covariant derivatives of the Bianchi identities for the R_{abcd} involved in (58). We can rewrite I_{Cov} [**?**] in a more concise form using **bivectors**, where we introduce the bivector-indices $\underline{A} := 12,\ \underline{B} := 31,\ \underline{C} := 23$. The integrability condition I_{Cov} can then be expressed as

$$\begin{aligned} I_{Cov} &= f^{(R)}_{\underline{AA};(33)} + f^{(R)}_{\underline{BB};(22)} + f^{(R)}_{\underline{CC};(11)} + 2(f^{(R)}_{\underline{AC};(13)} + f^{(R)}_{\underline{BC};(12)} + f^{(R)}_{\underline{AB};(23)}) \\ &= \sum_{x,y=1}^{3} \sum_{X,Y=1}^{3} f^{(R)}_{XY;(xy)} \; . \end{aligned} \tag{59}$$

Our new covariantly prolonged system is then given by

$$\begin{aligned} 0 &= f^{(R)}_{abcd} \\ 0 &= f^{(R)}_{abcd;e} \\ 0 &= I_{Cov} \; . \end{aligned} \tag{60}$$

It is $r(\mathcal{R}_2^{(1)}) = 6 + 19 = 25, r(\mathcal{R}_3^{(1)}) = 6 + 19 + 38 = 63$ as well as $r(\mathcal{M}_2^{(1)}) = 19, r(\mathcal{M}_3^{(1)}) = 38$ so that $\mathcal{M}_3^{(1)}$ coincides with the previous symbol for the prolongation involving partial derivatives. Again, it is $\dim(\mathcal{R}_2^{(2)}) = \dim(\mathcal{R}_2^{(1)}) = 55$. We find that the system (60) also consists of a system in involution and in addition to this also obeys general covariance. Therefore, we prefer (60) as a prolongation to a second-order system in involution for the Riemann-Lanczos problem in 3 dimensions and we state the final proposition:

Theorem 7 *Proposition*
The Riemann-Lanczos problem in 3 dimensions is not in involution. Its reduced characters (s'_0, s'_1, s'_2, s'_3) *are (14,8,7,3). The differential gauge conditions do not change the non-involutivity and they modify the reduced characters* (s'_0, s'_1, s'_2, s'_3) *only slightly to (17,8,7,0).*

The 3-dimensional Riemann-Lanczos problem becomes involutive after just one prolongation which is obtained by adding either of the integrability conditions (53) or (58) and now its Cartan characters $(\alpha_2^{(1)}, \alpha_2^{(2)}, \alpha_2^{(3)})$ *for* $\mathcal{R}_2^{(1)}$ *are (18,9,2).*

D. A Singular Solution for the 3-dimensional Reduced Gödel Spacetime

Lastly, we give a singular solution for the unprolonged 3-dimensional Gödel spacetime. In 3 dimensions, the reduced Gödel spacetime can be characterised by the following line element

$$ds^2 = a^2(dt^2 - dx^2 + \frac{1}{2}e^{2x}dy^2 + 2e^x dtdy)\,. \tag{61}$$

Here, we use the notation $x^1 := t, x^2 := x, x^3 := y$ for convenience. The Riemann-Lanczos equations together with the 3 differential gauge conditions are given explicitely in [?] and are very similar to the equations in 4 dimensions. Here, we want to find a solution which only depends on x leading to the *Ansatz*

$$\begin{aligned}
L_{txy} &= C_1 e^x\,, & L_{tyx} &= C_2 e^x\,,\\
L_{txt} &= C_3\,, & L_{xyy} &= C_7 e^{2x}\,,\\
L_{txx} &= C_4 e^{-x}\,, & L_{tyt} &= C_6\,,\\
L_{tyy} &= C_5 e^x\,, & L_{xyx} &= C_8\,,
\end{aligned}$$

where $C_1, ...C_8$ are arbitrary constants. When inserting this into the 6 independent Riemann Lanczos equations and the 3 differential gauge conditions, we obtain a solution which does not satisfy $L_{ty}{}^{s}{}_{;s} = 0$ [?] but when for instance choosing $C_5 = C_6 = 0$, we obtain the solution

$$\begin{aligned}
L_{txy} &= -\tfrac{a^2}{8}e^{(x)^2}\,, & L_{tyx} &= \tfrac{a^2}{8}e^x\,,\\
L_{txt} &= -\tfrac{a^2}{8}\,, & L_{xyy} &= \tfrac{3a^2}{16}e^{2x}\,.
\end{aligned}$$

Its singular solution manifold possesses 3-dimensional tangent spaces which are spanned by

$$\begin{aligned}
U &= \frac{\partial}{\partial t}\,,\\
V &= \frac{\partial}{\partial x} + \frac{a^2}{8}e^x\frac{\partial}{\partial L_{txy}} + \frac{a^2}{8}e^x\frac{\partial}{\partial L_{tyx}} + \frac{3a^2}{8}e^{2x}\frac{\partial}{\partial L_{xyy}}\\
&\quad + \frac{1}{3}e^x(\frac{5a}{16} - \frac{1}{2} - e^{2x}\frac{3a}{16})\frac{\partial}{\partial P_{txyx}} - \frac{1}{3}e^x(\frac{5a}{16} - \frac{1}{2} - e^{2x}\frac{3a}{16})\frac{\partial}{\partial P_{tyxx}}\\
&\quad + \frac{3}{4a}e^{2x}\frac{\partial}{\partial P_{xyyx}}\,,\\
Z &= \frac{\partial}{\partial y}\,.
\end{aligned} \tag{62}$$

We conclude that for this solution manifold $s_0 = s_0' = 3$ while all other characters vanish.

Conclusion

In 2 dimensions, the Riemann-Lanczos problem is very simple and we showed that it is always in involution. The general solution was given for both possible choices of signatures, Lorentzian and Euclidean.

In 3 dimensions, a prolongation becomes necessary to make it a system in involution. An integrability condition based on the derivatives of the Bianchi identities occurs when we use Janet-Riquier theory and introduce the second-order partial derivatives S_{abcde} as new jet coordinates. A singular solution for the unprolonged problem for the reduced Gödel spacetime is given.

Acknowledgements

Both authors wish to thank D Hartley and W M Seiler for valuable discussions as well as Prof L S Xanthis. We are grateful to Prof C Hoenselaers for pointing out references [**?**, **?**, **?**, **?**] to us.

A Gerber would like to thank the Swiss National Science Foundation (SNSF) and the Dr Robert Thyll-Dürr Foundation.

Appendix A: Calculations for 2 Dimensions

In this section we exhibit results and calculations which are necessary for the Riemann-Lanczos problem in 2 dimensions. First, $\delta_3, \cdots, \delta_6$ are given by

$$
\begin{aligned}
\delta_3 &= g^{11}(2\Gamma^1_{11} + \Gamma^2_{12}) + g^{12}(3\Gamma^1_{12} + \Gamma^2_{22}) + g^{22}\Gamma^1_{22}\,, \\
\delta_4 &= g^{12}(\Gamma^1_{11} + 3\Gamma^2_{12}) + g^{22}(\Gamma^1_{12} + 2\Gamma^2_{22}) + g^{11}\Gamma^2_{11}\,, \\
\delta_5 &= P_{1211}g^{11}_{\ ,1} + P_{1221}g^{12}_{\ ,1} + P_{1212}g^{12}_{\ ,1} + P_{1222}g^{22}_{\ ,1} - L_{121}\delta_{3,1} - L_{122}\delta_{4,1}\,, \\
\delta_6 &= P_{1211}g^{11}_{\ ,2} + P_{1221}g^{12}_{\ ,2} + P_{1212}g^{12}_{\ ,2} + P_{1222}g^{22}_{\ ,2} - L_{121}\delta_{3,2} - L_{122}\delta_{4,2}\,.
\end{aligned}
\tag{A.1}
$$

The polar matrices when the differential gauge condition is included then look like

equation	dx^1	dx^2	dL_{121}	dL_{122}	dP_{1211}	dP_{1212}	dP_{1221}	dP_{1222}
$df^{(R)}_{1212}$	α_{12121}	α_{12122}	$2(\Gamma^1_{12} - \Gamma^2_{22})$	$-2(\Gamma^1_{11} + \Gamma^2_{12})$	0	-2	2	0
$dL_{12}{}^{s}{}_{;s}$	δ_5	δ_6	$-\delta_3$	$-\delta_4$	g^{11}	g^{12}	g^{12}	g^{22}
K_{121}	$-P_{1211}$	$-P_{1212}$	1	0	0	0	0	0
K_{122}	$-P_{1221}$	$-P_{1222}$	0	1	0	0	0	0
$2(U\rfloor dK_{121})$	$-U_{1211}$	$-U_{1212}$	0	0	U^1	U^2	0	0
$2(U\rfloor dK_{122})$	$-U_{1221}$	$-U_{1222}$	0	0	0	0	U^1	U^2
$2(V\rfloor dK_{121})$	$-V_{1211}$	$-V_{1212}$	0	0	V^1	V^2	0	0
$2(V\rfloor dK_{122})$	$-V_{1221}$	$-V_{1222}$	0	0	0	0	V^1	V^2

,

where U forms a 1-dimensional integral element and $span\{U, V\}$ form a 2-dimensional one. The linear multipliers $A_1, \cdots, A_6$ and $B_1, \cdots, B_6$ are given by

$A_1 = \frac{1}{2}A_2(\frac{U^1}{U^2}g^{22} - g^{12})$, $\quad A_2 = \frac{V^2 - \frac{U^2}{U^1}V^1}{\delta_7}$,

$A_3 = A_2(\delta_3 - (\Gamma^1_{12} - \Gamma^2_{22})(\frac{U^1}{U^2}g^{22} - g^{12}))$, $A_4 = A_2(\delta_4 + (\Gamma^1_{11} + \Gamma^2_{12})(\frac{U^1}{U^2}g^{22} - g^{12}))$,

$A_5 = \frac{1}{U^1}(V^1 - g^{11}A_2)$, $\quad A_6 = -\frac{1}{U^2}g^{22}A_2$,

$B_1 = \frac{1}{2}B_2(g^{12} - \frac{U^2}{U^1}g^{11})$, $\quad B_2 = \frac{V^1 - \frac{U^1}{U^2}V^2}{\delta_8}$,

$B_3 = B_2(\delta_3 - (\Gamma^1_{12} - \Gamma^2_{22})(g^{12} - \frac{U^2}{U^1}g^{11}))$, $B_4 = B_2(\delta_4 + (\Gamma^1_{11} + \Gamma^2_{12})(g^{12} - \frac{U^2}{U^1}g^{11}))$,

$B_5 = -\frac{1}{U^1}g^{11}B_2$, $\quad B_6 = \frac{1}{U^2}(V^2 - g^{22}B_2)$,

where

$$\delta_7 = \delta_8 = 2g^{12} - \frac{U^2}{U^1}g^{11} - \frac{U^1}{U^2}g^{22} .$$

Appendix B: Calculations for 3 Dimensions

The quantity α_{abcde} for the Riemann-Lanczos problem in 3 dimensions is defined as

$$\begin{aligned}\alpha_{abcde} = {} & \Gamma^n_{ad,e}(L_{nbc} + L_{ncb}) + \Gamma^n_{bc,e}(L_{nad} + L_{nda}) - \Gamma^n_{ac,e}(L_{nbd} + L_{ndb}) \\ & -\Gamma^n_{bd,e}(L_{nac} + L_{nca}) .\end{aligned} \tag{B.1}$$

We obtain 18 different $\tilde{\alpha}_{abcde}$ for the 3-dimensional Riemann-Lanczos equations when using the full polar matrix, which are given as follows [?]
(for $e = 1, 2, 3$):

$$\begin{aligned}
\tilde{\alpha}_{1212e} = {} & R_{1212,e} + \alpha_{1212e} + 2(\Gamma^1_{12} + \Gamma^2_{22})P_{121e} - 2(\Gamma^1_{11} + \Gamma^2_{12})P_{122e} \\
& +2\Gamma^3_{22}P_{131e} + 2\Gamma^3_{11}P_{232e} + 2\Gamma^3_{12}P_{123e} - 4\Gamma^3_{12}P_{132e} \\
\tilde{\alpha}_{1313e} = {} & R_{1313,e} + \alpha_{1313e} + 2\Gamma^2_{33}P_{121e} + 2(\Gamma^1_{13} + \Gamma^3_{33})P_{131e} - 2(\Gamma^1_{11} + \Gamma^3_{13})P_{133e} \\
& -2\Gamma^2_{11}P_{233e} - 4\Gamma^2_{13}P_{123e} + 2\Gamma^2_{13}P_{132e} \\
\tilde{\alpha}_{2323e} = {} & R_{2323,e} + \alpha_{2323e} - 2\Gamma^1_{33}P_{122e} - 2\Gamma^1_{22}P_{133e} + 2(\Gamma^2_{23} + \Gamma^3_{33})P_{232e} \\
& -2(\Gamma^2_{22} + \Gamma^3_{23})P_{233e} + 2\Gamma^1_{23}P_{123e} + 2\Gamma^1_{23}P_{132e} \\
\tilde{\alpha}_{1213e} = {} & R_{1213,e} + \alpha_{1213e} + (\Gamma^1_{13} + 2\Gamma^2_{23})P_{121e} - \Gamma^2_{13}P_{122e} + (\Gamma^1_{12} + 2\Gamma^3_{23})P_{131e} \\
& -\Gamma^3_{12}P_{133e} - \Gamma^2_{11}P_{232e} - (\Gamma^1_{11} + 2\Gamma^2_{12} - \Gamma^3_{13})P_{123e} - (\Gamma^1_{11} + 2\Gamma^3_{13} - \Gamma^2_{12})P_{132e} \\
\tilde{\alpha}_{1223e} = {} & R_{1223,e} + \alpha_{1223e} - \Gamma^1_{23}P_{121e} + (2\Gamma^1_{13} + \Gamma^2_{23})P_{122e} + \Gamma^1_{22}P_{131e} - \Gamma^3_{22}P_{133e} \\
& -(\Gamma^2_{12} + 2\Gamma^3_{13})P_{232e} + \Gamma^3_{12}P_{233e} - (\Gamma^1_{12} + 2\Gamma^2_{22} + \Gamma^3_{23})P_{123e} \\
& +(2\Gamma^3_{23} + 2\Gamma^2_{22} - \Gamma^1_{12})P_{132e} \\
\tilde{\alpha}_{1323e} = {} & R_{1323,e} + \alpha_{1323e} - \Gamma^1_{33}P_{121e} + \Gamma^2_{33}P_{122e} + \Gamma^1_{23}P_{131e} - (2\Gamma^1_{12} + \Gamma^3_{23})P_{133e} \\
& +\Gamma^2_{13}P_{232e} - (\Gamma^3_{13} + 2\Gamma^2_{12})P_{233e} - (\Gamma^1_{12} + 2\Gamma^2_{23} - \Gamma^3_{33})P_{123e} \\
& +(\Gamma^1_{13} + 2\Gamma^3_{33} + \Gamma^2_{23})P_{132e}.
\end{aligned} \tag{B.2}$$

The polar matrix of $H(E^3)_x$ for the 3-dimensional Riemann-Lanczos problem can be written as

$$\begin{pmatrix} M_{01} & M_{02} & M_{03} \\ 0 & M_1 & 0 \end{pmatrix} .$$

M_{01} then is (for dx^1, dx^2, dx^3) the 6x3-matrix

$$M_{01} := \begin{pmatrix} R_{1212,1} + \alpha_{12121} & R_{1212,2} + \alpha_{12122} & R_{1212,3} + \alpha_{12123} \\ R_{1313,1} + \alpha_{13131} & R_{1313,2} + \alpha_{13132} & R_{1313,3} + \alpha_{13133} \\ R_{2323,1} + \alpha_{23231} & R_{2323,2} + \alpha_{23232} & R_{2323,3} + \alpha_{23233} \\ R_{1213,1} + \alpha_{12131} & R_{1213,2} + \alpha_{12132} & R_{1213,3} + \alpha_{12133} \\ R_{1223,1} + \alpha_{12231} & R_{1223,2} + \alpha_{12232} & R_{1223,3} + \alpha_{12233} \\ R_{1323,1} + \alpha_{13231} & R_{1323,2} + \alpha_{13232} & R_{1323,3} + \alpha_{13233} \end{pmatrix}$$

followed by the 8 columns (for the dL_{abc}) forming the 6x8-matrix M_{02} which is

$$\begin{pmatrix} 2(\Gamma^1_{12} + \Gamma^2_{22}) & -2(\Gamma^1_{11} + \Gamma^2_{12}) & 2\Gamma^3_{22} & 0 & 2\Gamma^3_{11} & 0 & 2\Gamma^3_{12} & -4\Gamma^3_{12} \\ 2\Gamma^2_{33} & 0 & 2(\Gamma^1_{13} + \Gamma^3_{33}) & -2(\Gamma^1_{11} + \Gamma^3_{13}) & 0 & -2\Gamma^2_{11} & -4\Gamma^2_{13} & 2\Gamma^2_{13} \\ 0 & -2\Gamma^1_{33} & 0 & -2\Gamma^1_{22} & 2(\Gamma^2_{23} + \Gamma^3_{33}) & -2(\Gamma^2_{22} + \Gamma^3_{23}) & 2\Gamma^1_{23} & 2\Gamma^1_{23} \\ \Gamma^1_{13} + 2\Gamma^2_{23} & -\Gamma^2_{13} & \Gamma^1_{12} + 2\Gamma^3_{23} & -\Gamma^3_{12} & -\Gamma^2_{11} & \Gamma^3_{11} & -(\Gamma^1_{11} + 2\Gamma^2_{12} - \Gamma^3_{13}) & -(\Gamma^1_{11} + 2\Gamma^3_{13} - \Gamma^2_{12}) \\ -\Gamma^1_{23} & 2\Gamma^1_{13} + \Gamma^2_{23} & \Gamma^1_{22} & -\Gamma^3_{22} & -(\Gamma^2_{12} + 2\Gamma^3_{13}) & \Gamma^3_{12} & -(\Gamma^1_{12} + 2\Gamma^2_{22} + \Gamma^3_{23}) & 2\Gamma^3_{23} + \Gamma^2_{22} - \Gamma^1_{12} \\ -\Gamma^1_{33} & \Gamma^2_{33} & \Gamma^1_{23} & -(2\Gamma^1_{12} + \Gamma^3_{23}) & \Gamma^2_{13} & -(\Gamma^3_{13} + 2\Gamma^2_{12}) & -(-\Gamma^1_{13} + 2\Gamma^2_{23} - \Gamma^3_{33}) & \Gamma^1_{13} + 2\Gamma^3_{33} + \Gamma^2_{23} \end{pmatrix}$$

then followed by the 24 columns (for the dP_{abcd}) producing the 6x24-matrix

$$M_{03} := \begin{pmatrix} 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}.$$

M_1 consists of a 32x35-matrix which can be split into

$$M_1 := \begin{pmatrix} M(P) & 1 & 0 \\ M(U) & 0 & N(U) \\ M(V) & 0 & N(V) \\ M(Z) & 0 & N(Z) \end{pmatrix}.$$

M(P) is the 3x8-matrix

$$M(P) := \begin{pmatrix} -P_{1211} & -P_{1212} & -P_{1213} \\ -P_{1221} & -P_{1222} & -P_{1223} \\ -P_{1311} & -P_{1312} & -P_{1313} \\ -P_{1331} & -P_{1332} & -P_{1333} \\ -P_{2321} & -P_{2322} & -P_{2323} \\ -P_{2331} & -P_{2332} & -P_{2333} \\ -P_{1231} & -P_{1232} & -P_{1233} \\ -P_{1321} & -P_{1322} & -P_{1323} \end{pmatrix}.$$

M(U) is the 3x8-matrix

$$M(U) := \begin{pmatrix} -U_{1211} & -U_{1212} & -U_{1213} \\ -U_{1221} & -U_{1222} & -U_{1223} \\ -U_{1311} & -U_{1312} & -U_{1313} \\ -U_{1331} & -U_{1332} & -U_{1333} \\ -U_{2321} & -U_{2322} & -U_{2323} \\ -U_{2331} & -U_{2332} & -U_{2333} \\ -U_{1231} & -U_{1232} & -U_{1233} \\ -U_{1321} & -U_{1322} & -U_{1323} \end{pmatrix}.$$

N(U) then is the 8x24-matrix

$$
\begin{pmatrix}
U^1 & U^2 & U^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & U^1 & U^2 & U^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & U^1 & U^2 & U^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U^1 & U^2 & U^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U^1 & U^2 & U^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U^1 & U^2 & U^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U^1 & U^2 & U^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & U^1 & U^2 & U^3
\end{pmatrix}.
$$

M(V) and M(Z) are the same as M(U) only for the Vessiot vector fields V and Z. In the same way N(V) and N(Z) are the same as N(U).

Next, we give the linear multipliers creating a linear combination for the reduced polar matrix.

$$A_1 = \tfrac{\delta_1}{2} B_1 \,, \qquad A_2 = \tfrac{\delta_2^2}{2\delta_1} B_1 \,, \qquad A_3 = -\tfrac{\delta_4^2 V^2}{2\delta_3^2 V^1} B_1 \,,$$

$$A_4 = \delta_2 B_1 \,, \qquad A_5 = \tfrac{V^2 \delta_4}{V^1} B_1 \,, \qquad A_6 = \tfrac{V^2 \delta_2 \delta_4}{V^1 \delta_1} B_1 \,,$$

$$B_1 = B_1(arbitrary) \,, \qquad B_2 = \tfrac{V^2}{V^1} B_1 \,, \qquad B_3 = \tfrac{\delta_2}{\delta_1} B_1 \,,$$

$$B_4 = \tfrac{V^3 \delta_2}{V^2 \delta 1} B_1 \,, \qquad B_5 = -\tfrac{V^2 \delta_4}{V^1 \delta_3} B_1 \,, \qquad B_6 = -\tfrac{V^3 \delta_4}{V^2 \delta_3} B_1 \,,$$

$$B_7 = \tfrac{V^2}{V^1}(\tfrac{\delta_2}{\delta_1} - 2\delta_4) B_1 \,, \qquad B_8 = -\tfrac{1}{\delta_3}(\delta_2 + \delta_4) B_1 \,,$$

$$C_1 = -\tfrac{U^1}{V^1} B_1 \,, \qquad C_2 = -\tfrac{U^2}{V^1} B_1 \,, \qquad C_3 = -\tfrac{U^1 \delta_2}{V^1 \delta_1} B_1 \,,$$

$$C_4 = -\tfrac{U^3 \delta_2}{V^1 \delta_1} B_1 \,, \qquad C_5 = \tfrac{U^2 \delta_4}{V^1 \delta_3} B_1 \,, \qquad C_6 = \tfrac{U^3 \delta_4}{V^1 \delta_3} B_1 \,,$$

$$C_7 = -\tfrac{1}{V^1}(\delta_2 + U^1 V^2 \tfrac{\delta_2}{\delta_1} - 2\tfrac{V^2}{V^1} U^1 \delta_4) B_1, \; C_8 = (\tfrac{\delta_4}{V^1} + \tfrac{U^2 \delta_4}{V^2 \delta_3} + \tfrac{U^2 \delta_2}{V^2 \delta_3}) B_1,$$

where here

$$\delta_1 = U^2 - V^2 \tfrac{U^1}{V^1} \,, \quad \delta_2 = U^3 - V^3 \tfrac{U^1}{V^1} \,, \quad \delta_3 = U^1 - V^1 \tfrac{U^2}{V^2} \,,$$

$$\delta_4 = U^3 - V^3 \tfrac{U^2}{V^2} \,, \quad \delta_5 = U^1 - V^1 \tfrac{U^3}{V^3} \,, \quad \delta_6 = U^2 - V^2 \tfrac{U^3}{V^3} \,,$$

and

$$\delta_3 = -\frac{V^1}{V^2}\delta_1 \,, \quad \delta_5 = -\frac{V^1}{V^3}\delta_2 \,, \quad \delta_6 = -\frac{V^2}{V^3}\delta_4 \,.$$

For the special choice $B_1 = U^1 V^1 V^2$, we obtain a similar kind of identity as the one mentioned in (23) in [?] for the 4-dimensional case but not with exactly the same multipliers because we incorporated the cyclic conditions (1) as well.

Appendix C: A Coordinate Transformation for the 3-dimensional Case

The B_i in (51) are the parts which do not involve any term of the form V_{abc3} and are given by

$$
\begin{aligned}
B_1 &= a_{21}V_{1211} + a_{22}V_{1212} - a_{11}V_{1331} - a_{12}V_{1222} \\
B_2 &= a_{31}V_{1311} + a_{32}V_{2322} - a_{11}V_{1331} - a_{12}V_{2332} \\
B_3 &= a_{31}V_{2321} + a_{32}V_{2322} - a_{21}V_{2331} - a_{22}V_{2332} \\
B_4 &= a_{31}V_{1211} + a_{32}V_{1212} - a_{11}V_{1231} - a_{22}V_{1232} + a_{21}V_{1311} + a_{22}V_{1312} \\
&\quad - a_{11}V_{1321} - a_{12}V_{1232} \\
B_5 &= a_{31}V_{1221} + a_{32}V_{1222} - 2a_{21}V_{1231} - 2a_{22}V_{1232} + a_{21}V_{1321} + a_{22}V_{1322}
\end{aligned}
$$

$$
\begin{aligned}
&-a_{11}V_{2321} - a_{12}V_{2322} \\
B_6 &= 2a_{31}V_{1321} + 2a_{32}V_{1322} - a_{21}V_{1331} - a_{22}V_{1332} - a_{31}V_{1231} - a_{32}V_{1232} \\
&-a_{11}V_{2331} - a_{12}V_{2332}\,. \qquad \text{(C.1)}
\end{aligned}
$$

The function $f(V_{abc1}, V_{abc2})$ for $V_{2332} = f(V_{abc1}, V_{abc2})$ is given by

$$
\begin{aligned}
V_{2332} = &\frac{a_{23}}{a_{23}a_{12} - a_{22}a_{13}}[V_{1211}(\frac{a_{31}a_{33}}{a_{13}} - \frac{a_{21}{a_{33}}^2}{a_{13}a_{23}}) + V_{1212}(\frac{a_{33}a_{33}}{a_{13}} - \frac{a_{22}{a_{33}}^2}{a_{13}a_{23}}) \\
&+V_{1221}(\frac{a_{11}{a_{33}}^2}{a_{13}a_{23}} - \frac{a_{31}a_{33}}{a_{23}}) + V_{1222}(\frac{a_{12}{a_{33}}^2}{a_{13}a_{23}} - \frac{a_{32}a_{33}}{a_{23}}) + V_{1311} \\
&(\frac{a_{21}a_{33}}{a_{13}} - \frac{a_{21}a_{33}}{a_{13}}) + V_{1312}(\frac{a_{22}a_{33}}{a_{13}} - \frac{a_{23}a_{33}}{a_{23}}) + V_{1331}(\frac{a_{11}a_{33}}{a_{13}} - a_{21}) \\
&+V_{1332}(a_{12} - a_{22}) + V_{2321}(\frac{a_{11}a_{33}}{a_{23}} - \frac{a_{13}a_{31}}{a_{23}}) + V_{2322}(\frac{a_{12}a_{33}}{a_{23}} - \frac{a_{13}a_{32}}{a_{23}}) \\
&+V_{2331}(\frac{a_{13}a_{21}}{a_{23}} - a_{11}) + V_{1231}(2\frac{a_{21}a_{33}}{a_{23}} - \frac{a_{11}a_{33}}{a_{23}} - a_{31}) + V_{1232} \\
&(2\frac{a_{22}a_{33}}{a_{23}} - \frac{a_{12}a_{33}}{a_{13} - a_{32}}) + V_{1321}(2a_{31} - \frac{a_{11}a_{33}}{a_{13}} - \frac{a_{33}a_{21}}{a_{23}}) \\
&+V_{1322}(2a_{32} - \frac{a_{12}a_{33}}{a_{13}} - \frac{a_{22}a_{33}}{a_{23}})\,. \qquad \text{(C.2)}
\end{aligned}
$$

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[1] All monomials which are not in the set of principal monomials or in its extension by multiplicative variables correspond to parametric derivatives.

[2] Note that if a system of partial differential equations is translated into an exterior differential system in the appropriate way, then we have the correspondence $s_k = \alpha_q^{(k)}$ as long as it is in involution. We refer the reader to [**?**, **?**] for further details concerning *involution*.